Chapter 4
The DDA Method

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ABSTRACT

"DDA" stands for "Discontinuous Deformation Analysis", suggesting that the displacement field of the analyzed domain shows abrupt changes on the element boundaries in the model. This chapter introduces the theoretical fundaments of DDA: mechanical characteristics of the elements together with the basic degrees of freedom, contact behavior, the equations of motion and their numerical integration with the help of Newmark's beta-method taking into account contact creation, loss and sliding with the help of an open-close iteration technique. Finally, a short overview on practical and scientific applications for masonry structures is given.

INTRODUCTION

DDA is a numerical technique to simulate the mechanical behavior of a collection of polyhedral blocks. "DDA" stands for "Discontinuous Deformation Analysis", suggesting that the displacement field shows abrupt changes on the element boundaries in the model. (This is a fundamental difference from a finite element model where continuity conditions have to be satisfied at the coinciding nodes of neighboring elements.)

The first DDA model was published by Gen-Hua Shi in his PhD dissertation in 1988 and in a journal paper Shi, 1992. That code was a two-dimensional model, with uniform-strain polygonal elements, having undeformable contacts that transmitted normal and tangential forces between the elements. Shi originally suggested his model for the analysis of fractured rocks. In the next decades several versions of DDA were published: three-dimensional models were born (Shi, 2001), different element shapes were applied (e.g. spheres, Zhao et al, 2000), higher-order displacement fields inside the elements were used (MacLaughlin, 1997) etc. The method has been widely applied for the analysis of masonry structures like arches, columns or stone bridges (Thavalingam et al, 2001; Scheldt et al, 2002; Kamai et al, 2005; Kamai & Hatzor, 2008; Rizzi et al, 2014 etc).

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This chapter focuses on DDA with polyhedral elements. The basic degrees of freedom and the element behavior are introduced in Section 2. Section 3 focuses on the contact behavior. Section 4 summarizes the quantities in the equations of motion, and Section 5 introduces how the time integration is done: the main idea of the calculation of a single time step is explained. Then in Section 6 applications of DDA to masonry problems are shown.

THE ELEMENTS

The elements in Shi's model have arbitrary polyhedral shapes. Each element has a reference point, which is the centre of gravity of the element. Denote the position of the reference point of element p by (x^p, y^p, z^p).

The element can translate and rotate as a rigid body (6 degrees of freedom); and in addition to that, a uniform deformation field (characterized by the usual small strain tensor) belongs to the element (another 6 degrees of freedom). The element has altogether 12 degrees of freedom, and the 12 unknown kinematic characteristics are summarized into the generalized displacement vector of the element:

$$\mathbf{u}^p = \begin{bmatrix} u_x^p \\ u_y^p \\ u_z^p \\ \varphi_x^p \\ \varphi_y^p \\ \varphi_z^p \\ \varepsilon_x^p \\ \varepsilon_y^p \\ \varepsilon_z^p \\ \gamma_{yz}^p \\ \gamma_{zx}^p \\ \gamma_{xy}^p \end{bmatrix}$$

Assuming small displacements, the translations of an arbitrary (x, y, z) point on the element can uniquely be determined with the help of the $\mathbf{u}^p$ generalized displacement vector of the element:

$$\begin{bmatrix} u_x(x,y,z) \\ u_y(x,y,z) \\ u_z(x,y,z) \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & (z-z^p) & -(y-y^p) & (x-x^p) & 0 & 0 & 0 & \frac{(z-z^p)}{2} & \frac{(y-y^p)}{2} \\ 0 & 1 & 0 & -(z-z^p) & 0 & (x-x^p) & 0 & (y-y^p) & 0 & \frac{(z-z^p)}{2} & 0 & \frac{(x-x^p)}{2} \\ 0 & 0 & 1 & (y-y^p) & -(x-x^p) & 0 & 0 & 0 & (z-z^p) & \frac{(y-y^p)}{2} & \frac{(x-x^p)}{2} & 0 \end{bmatrix} \begin{bmatrix} u_x^p \\ u_y^p \\ u_z^p \\ \varphi_x^p \\ \varphi_y^p \\ \varphi_z^p \\ \varepsilon_x^p \\ \varepsilon_y^p \\ \varepsilon_z^p \\ \gamma_{yz}^p \\ \gamma_{zx}^p \\ \gamma_{xy}^p \end{bmatrix}$$

or shortly:

$$\mathbf{u}(x,y,z) = \mathbf{T}^p(x,y,z)\ \mathbf{u}^p\ .$$

Note that these relations are valid for small incremental rotations only (the issue of large rotations will be discussed at the end of Section 2).

This translation vector is particularly important for those points on the element boundary where a contact is formed with a neighbouring element. The difference between the translations of the two material points belonging to the two elements that form the contact is considered the *contact deformation*. This relative translation determines the forces transmitted through the contact, according to the contact constitutive relations.

Note that the role of matrix $\mathbf{T}$ is similar to that of the transition matrix in FEM; the difference is that in DDA the deformation of the element is also taken into account.

The forces acting on the elements are either external (like gravity, or velocity-dependent drag forces) or contact forces expressed by the neighbouring elements. Consider element p, and consider all the forces acting on it, either external or contact forces. Reducing them to the reference point of the element 6 scalar componnts are received. In addition, corresponding to the uniform strain, a uniform stress field is assigned to element p which has 6 more scalar characteristics. So, altogether 12 components form the $\mathbf{f}^p$ generalized reduced force vector of element p:

$$\mathbf{f}^p = \begin{bmatrix} f_x^p \\ f_y^p \\ f_z^p \\ m_x^p \\ m_y^p \\ m_z^p \\ V^p\sigma_x^p \\ V^p\sigma_y^p \\ V^p\sigma_z^p \\ V^p\tau_{yz}^p \\ V^p\tau_{zx}^p \\ V^p\tau_{xy}^p \end{bmatrix}$$

It is important to emphasize that the uniform stress field in the element does not hold a direct relationship with the contact forces: the usual static boundary condition is not valid for the stress field at the boundary of a DDA element and the contact forces acting on it.

Note that if an element performing an incremental generalized displacement $d\mathbf{u}^p$ is acted upon by a generalized reduced force $\mathbf{f}^p$, the scalar product $(d\mathbf{u}^p)^T\mathbf{f}^p$ means work increment. (This is the reason to multiply the lower 6 components of $\mathbf{f}^p$ with the V^p volume of the element.)

Consider now the force $\mathbf{F}^c$ acting on the analysed element in point c. This is reduced to the reference point with the help of the transpose of the matrix $\mathbf{T}$, so the term $(\mathbf{T}^T\mathbf{F}^c)$ is added to other forces when compiling the reduced force $\mathbf{f}^p$:

$$\mathbf{T}^{pT}\mathbf{F}^c = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ 0 & -(z-z^p) & (y-y^p) & \\ (z-z^p) & 0 & -(x-x^p) & \\ -(y-y^p) & (x-x^p) & 0 & \\ (x-x^p) & 0 & 0 & \\ 0 & (y-y^p) & 0 & \\ 0 & 0 & (z-z^p) & \\ 0 & \frac{(z-z^p)}{2} & \frac{(y-y^p)}{2} & \\ \frac{(z-z^p)}{2} & 0 & \frac{(x-x^p)}{2} & \\ \frac{(y-y^p)}{2} & \frac{(x-x^p)}{2} & 0 & \end{bmatrix} \begin{bmatrix} F_x^c \\ F_y^c \\ F_z^c \end{bmatrix}$$

Independently of whether a contact force $\mathbf{F}^c$ is considered or any other concentrated external effect, the model of Shi gives the same simplified relation between forces and stresses as seen in the last six rows of the above relation. This approximately means as if calculating the average uniform stress caused by all the forces which act on the element.

In the practical analysis of fractured rock masses the initial self-stress must be taken into consideration for the reliable modelling. This can be done with the lower six components of $\mathbf{f}^p$. (The details of how to compile $\mathbf{f}^p$ from the different mechanical effects can be found in Shi, 2001.)

In the original DDA model the constitutive relations represent a linearly elastic, isotropic material. However, in principle any other kind of constitutive relations can be applied in the model, assuming that a potential energy can be assigned to the deformations of the elements.

DDA models with higher order strain fields (e.g. MacLaughlin, 1997) work in the same manner, consequently higher order stress fields appear, which means more kinematic and static variables belonging to the elements.

Another possibility to have a non-uniform strain field in the elements is to subdivide them into sub-blocks. This technique was applied by Pérez-Aparicio et al (2013). They successfully simulated the collapse of covered arches with different geometries and loads. The subdivision into sub-blocks allows for a more reliable prediction of collapses due to material failure.

Jiang and Zheng (2015) analysed an issue observed in the original version of DDA when the blocks undero large rotations. For a small rotation φ, the first-order approximations $\cos\varphi \approx 1$ and $\sin\varphi \approx \varphi$ (applied above) are acceptable, but at large rotations they lead to cumulative errors and as a result, a false volume expansion occurs that was observed by several authors (see Jiang and Zheng, 2015 for an overview). To avoid this error, Jiang and Zheng (2015) suggested to fix a local coordinate frame to each block, which moves and rotates together with the block. After the open-close iteration procedure determines the incremental motions of the block done during the analysed time step, the strain increments are transformed to the local coordinate frame and added to the accumulated strain of the block. Using this improvement, the authors succeeded to avoid the occurrence of false volume expansion regardless of the magnitude of block rotation.

THE CONTACTS

When two elements move into a position where they overlap each other, a contact is formed. This is represented in the mechanical calculations by the two material points forming the first entrance. Overlapping is resisted by penalty functions in the original DDA method: a compressional contact force occurs whose magnitude depends linearly on the depth of the overlap (it is practically equivalent to an elastic, no-tension contact model), and tangential forces are treated in the same way. Coulomb-friction gives a limit to the tangential force magnitude. More specifically, assume that the contact is formed by the material points pc and qc on the boundary of elements p and q. In terms of the increment of the generalized displacement vectors, $d\mathbf{u}^p$ and $d\mathbf{u}^q$, the increment of the relative translation vector at the contact can be expressed as $\mathbf{T}^q(x^{qc}, y^{qc}, z^{qc})\, d\mathbf{u}^q - \mathbf{T}^p(x^{pc}, y^{pc}, z^{pc})\, d\mathbf{u}^p$. The mechanical model of the contact behavior specifies how the contact force components are calculated from the relative translation components. In the simplest case constant normal and tangential stiffnesses (corresponding to constant penalty parameters) are applied, and the contact starts to slide when the tangential force reaches the Coulomb-limit.

Several DDA simulations have been performed on the sliding motion of a single block on an inclined surface (e.g. Doolin and Sitar, 2002; Tsesarsky et al, 2005; see an overview in MacLaughlin and Doolin, 2006). Tsesarsky et al (2005) recommended improving DDA by incorporating displacement-dependent friction, and Bakun-Mazor et al (2012) found that velocity-dependent friction should also be included.

In reality a tensional resistance can also occur between masonry blocks, and this may also be important to incorporate into the model. Seemingly, cohesion can easily be included in the contact model: if the separation or the relative tangential displacement of the two material points forming the contact exceeds a threshold, the contact is deleted; but until then, a tensional and a shear force occur proportionally to the gap size and the relative tangential displacement respectively. However, the performance of this simple model is questionable. Wang et al (2013) introduced a displacement-dependent shear strength model for the contacts, and illustrated the advantages on landslide simulations. In an example of a block sliding on a plane they pointed out that while the original DDA differed up to a 100% from the analytical solution, the displacement-dependent contact model gave an excellent coincidence with the theoretical results.

THE EQUATIONS OF MOTION

The theoretical basis of the derivation of the equations of motion in DDA is the potential energy function containing terms for element deformability, contact deformability (or penalty functions), D'Alembert forces, concentrated and body loads, initial stress and prescribed displacement history. The stationary theorem of the potential energy states that the first derivatives of the potential energy (according to every component of the generalized displacement vectors of the elements) have to be zero. This condition yields the system of dynamic equations, which can be arranged to the following form ("generalized equations of motion"):

$$\mathbf{M}\mathbf{a}(t) + \mathbf{C}\mathbf{v}(t) + \mathbf{K}(t)\mathbf{u}(t) = \mathbf{f}(t)$$

where the generalized displacement vector and reduced force vector for the total system containing N elements are: $\mathbf{u}(t) = \begin{bmatrix} \mathbf{u}^1(t) \\ \mathbf{u}^2(t) \\ \vdots \\ \mathbf{u}^N(t) \end{bmatrix}$; $\mathbf{f}(t) = \begin{bmatrix} \mathbf{f}^1(t, u(t), v(t)) \\ \mathbf{f}^2(t, u(t), v(t)) \\ \vdots \\ \mathbf{f}^N(t, u(t), v(t)) \end{bmatrix}$.

The generalized velocities and accelerations are

$$\mathbf{v}(t) = \frac{d\mathbf{u}(t)}{dt}, \quad \mathbf{a}(t) = \frac{d^2\mathbf{u}(t)}{dt^2}$$

The exact derivations can be found in Shi, 2001, detailed at a computer-programming level.

Corresponding to every element, there are 12 scalar equations in the generalized equations of motion. The first 3 equations set the link between translational accelerations of the element and the first 3 components in the reduced force vector. In the second 3 equations the rotational accelerations and the

three moment components are related. The last 6 equations express the relations between stresses and the external loads acting on the element.

The matrix of inertia, **M**, has a block-diagonal structure:

$$\mathbf{M} = \begin{bmatrix} \mathbf{M}^1 & & & \\ & \mathbf{M}^2 & & \\ & & \ddots & \\ & & & \mathbf{M}^N \end{bmatrix},$$

where the $\mathbf{M}^p$ block belonging to the p-th element can be expressed from the $\mu(x, y, z)$ density function of the element, with the help of matrix **T** which was seen above:

$$\mathbf{M}^p = \int_{V^p} \mathbf{T}^T(x, y, z) \cdot \mathbf{T}(x, y, z) \cdot \mu(x, y, z)\, dV$$

(again, the details of the derivations can be found in Shi, 2001).

TIME INTEGRATION

The time integration is performed with the help of Newmark's β–method. The motion of the elements along time is simulated in a series of small finite time intervals.

Consider the time interval (t_i, t_{i+1}). At its beginning, t_i, the state of the system is known: the vectors $\mathbf{u}_i$, $\mathbf{v}_i$, $\mathbf{f}(t_i, \mathbf{u}_i, \mathbf{v}_i)$ are known, and they satisfy the generalized equations of motion. The aim of the calculations is to ensure that the equations of motion

$$\mathbf{0} = \mathbf{f}(t, \mathbf{u}(t), \mathbf{v}(t)) - \mathbf{M} \cdot \mathbf{a}(t) - \mathbf{C}\mathbf{v}(t) - \mathbf{K}(t)\mathbf{u}(t)$$

would be satisfied also at t_{i+1}:

$$\mathbf{0} = \mathbf{f}(t_{i+1}, \mathbf{u}_{i+1}, \mathbf{v}_{i+1}) - \mathbf{M}_{i+1} \cdot \mathbf{a}_{i+1} - \mathbf{C}_{i+1}\mathbf{v}_{i+1} - \mathbf{K}_{i+1}\Delta\mathbf{u}_{i+1}$$

where the aim is to find that $\Delta\mathbf{u}_{i+1} = \mathbf{u}_{i+1} - \mathbf{u}_i$ generalized displacement increment which takes the system to the end of the timestep into a state where the equations of motion are satisfied.

In addition to $\Delta\mathbf{u}_{i+1}$ the unknown $\mathbf{v}_{i+1}$ and $\mathbf{a}_{i+1}$ are also contained in the generalized equations of motion. According to the Newmark β-method with parameters $\beta = 1/2$ and $\gamma = 1$, approximate these unknowns in terms of $\Delta\mathbf{u}_{i+1}$ in the following way:

$$\mathbf{a}_{i+1} = \frac{1}{\Delta t^2 / 2}\left(\Delta\mathbf{u}_{i+1} - \Delta t \cdot \mathbf{v}_i\right)$$
$$\mathbf{v}_{i+1} = \mathbf{v}_i + \Delta t \cdot \mathbf{a}_{i+1}$$

The mechanical meaning of these approximations is that the acceleration $\mathbf{a}_{i+1}$ is valid in the whole (t_i, t_{i+1}) interval, independently of the accelerations at the end of the previous time interval, i.e. at t_i. (Note that for $\beta = 1/2$ and $\gamma = 1$ the Newmark-method is unconditionally stable.)

Inserting the above expressions for $\mathbf{a}_{i+1}$ and $\mathbf{v}_{i+1}$ into the equations of motion, $\Delta\mathbf{u}_{i+1}$ remains the only unknown. A new notation is introduced:

$$\mathbf{f}(t_{i+1}, \mathbf{u}_{i+1}, \mathbf{v}_{i+1}) - \mathbf{M}_{i+1} \cdot \mathbf{a}_{i+1} - \mathbf{C}_{i+1}\mathbf{v}_{i+1} - \mathbf{K}_{i+1}\Delta\mathbf{u}_{i+1} := \mathbf{r}(t_{i+1}, \Delta\mathbf{u}_{i+1})$$

which means that the aim is to find that$\Delta\mathbf{u}_{i+1}$ which makes $\mathbf{r}$ zero at t_{i+1}. The vector $\mathbf{r}(t, \Delta\mathbf{u})$ will be called "residual", and its magnitude expresses the error according to which the equations of motion are violated.

In DDA the unknown $\Delta\mathbf{u}_{i+1}$ is calculated with the help of the Newton-Raphson iteration method where the Jacobian matrix of $\mathbf{r}(t, \Delta\mathbf{u})$ by $\Delta\mathbf{u}$ is needed:

$$\mathcal{K}(t, \Delta\mathbf{u}) = \frac{d\mathbf{r}(t, \Delta\mathbf{u})}{d\Delta\mathbf{u}}$$

This matrix is the sum of the usual stiffness matrix based on the contact stiffnesses, and of other terms due to the inertia, the stiffnesses of the material of the elements, and additional terms taking into account effects like self stress etc. In his publications Shi gave these derivations in detail (e.g. Shi, 2001), so the calculation of the Jacobian is straightforward. Hence, the residual for a given t and $\Delta\mathbf{u}$ can uniquely be calculated – the interested reader should consult the literature for the exact details.

Focus now on how to determine the unknown $\Delta\mathbf{u}_{i+1}$. Let its first approximation be a zero vector, and for the beginning of the calculations apply the already known position, velocity and internal forces of the system valid at t_i:

$$\Delta\mathbf{u}_{i+1}^{(0)} := \mathbf{0}\,.$$

If $\mathbf{r}(t_{i+1}, \Delta\mathbf{u}_{i+1}^{(0)}) = \mathbf{0}$ were valid, the calculation would be ready: the equations of motion would be satisfied. However, typically this is not the case, so a better approximation can be produced as

$$\Delta\mathbf{u}_{i+1}^{(1)} := \Delta\mathbf{u}_{i+1}^{(0)} - \mathcal{K}(t_{i+1}, \Delta\mathbf{u}_{i+1}^{(0)})^{-1} \cdot \mathbf{r}(t_{i+1}, \Delta\mathbf{u}_{i+1}^{(0)})\,.$$

Check now whether $\mathbf{r}(t_{i+1}, \Delta\mathbf{u}_{i+1}^{(1)})$ is sufficiently small. The iteration can be terminated if its norm is below a pre-defined threshold value; otherwise, a next approximation has to be prepared, according to the general formula:

$$\Delta\mathbf{u}_{i+1}^{(k+1)} := \Delta\mathbf{u}_{i+1}^{(k)} - \mathcal{K}(t_{i+1}, \Delta\mathbf{u}_{i+1}^{(k)})^{-1} \cdot \mathbf{r}(t_{i+1}, \Delta\mathbf{u}_{i+1}^{(k)})\,.$$

The iteration is continued until $\left|\mathbf{r}(t_{i+1}, \Delta\mathbf{u}_{i+1}^{(k+1)})\right|$ decreases below the threshold set by the user. (In some methods the norm $\left|\Delta\mathbf{u}_{i+1}^{(k+1)} - \Delta\mathbf{u}_{i+1}^{(k)}\right|$ is also checked, which is particularly useful in the case of nearly-singular $\mathcal{K}$.)

After finding $\Delta\mathbf{u}_{i+1}$ with sufficient accuracy, the next task is to check whether the topology of the system remains unchanged, or the displacements led to contacts being lost, created or slipped. If an overlap is found between two elements not in contact at the beginning of the timestep, DDA returns to the beginning of the timestep, puts a compressional spring between the two elements, and the calculation of the timestep is repeated. Similarly, if a contact is lost because of the calculated displacements, the contact is removed and the calculation of the timestep is repeated without that contact. If the friction limit is exceeded by a tangential force in a contact, then the timestep is repeated by keeping the shear force at the friction limit, and setting the shear stiffness to zero.

Obviously, during the re-calculation of a timestep with a modified contact, other contacts may be modified or created. In this case the calculation is repeated again and again ("open-close iterations"), until no change is experienced any more. If convergence is not attained within a specified number of iterations (typically six to eight), the time step is reduced (usually to its one third), and the analysis for that time step is repeated using the reduced time step. This is a recursive process, meaning if convergence is not attained with the reduced time step, the time step is further reduced (usually to one third again, i.e., 1/9 of the original time step). This recursive process increases the number of matrix iterations required for each time step and decreases the time step length used over the course of an analysis. The increase in computational time involved in the implicit method is not merely related to the cost of a single matrix inversion per time step. Multiple matrix inversions may be required during each time step, especially in systems where a large number of contacts are breaking and forming.

After finding the new state which satisfies the equations of motion as well as all conditions related to the contact behaviour, the next timestep can be analysed.

The most important parameters affecting the numerical behaviour of DDA are the contact normal stiffness, the time step length and the applied damping (if any). Yagoda-Biran and Hatzor (2016) discussed the existing suggestions previously published in the literature, and added their own experiences. They found, for instance, that depending on the actual physical features of the analysed problem, the value of the contact stiffness should be about 2-4 orders of magnitude lower than the product of the Young modulus of the intact blocks and the average block size. The interested reader can find the details in Yagoda-Biran and Hatzor (2016).

Regarding the behaviour of the open-close iteration technique, Khan (2010) and Khan et al (2010) compared the DDA solution of a 2D slope sliding problem with a solution given by UDEC (where an explicit time integration is used). It was found that DDA may require several times or orders of magnitude longer computational time than UDEC. This is partly because of the open-close iterations decreasing the time step length. Other reasons (e.g. type of the applied equation solver) are also extensively discussed by Khan (2010).

It has to be emphasized that the convergence of the DDA method has not been proven yet. With constant topology the applied parameters of the Newmark method ensure numerical stability, but if open-close iterations have to be performed, convergence cannot be ensured any more.

Finally, a short comparison is given between DDA and the commercial code 3DEC which is probably the most popular discrete element software in practical masonry analysis (see Chapter 3 of the present book):

- In 3DEC the *elements* are either perfectly rigid, or subdivided into tetrahedral finite elements; in DDA the elements are uniform-strain blocks (or higher order in some versions) so a strongly varying stress field within a block cannot reliably be described in DDA.
- The *basic unknowns* in 3DEC are the three translations and three rotation components for rigid blocks, or the nodal translations for tetrahedral subdivisions; in DDA the uniform-strain blocks have 12 degrees of freedom (translations, rotations and strain components).
- 3DEC applies an *explicit* time integration technique (the method of central differences), while DDA is based on an *implicit* method (Newmark's β–method).
- From numerical point of view, 3DEC is only conditionally stable hence the *time step length* is very limited; in DDA the implicit nature of the integration allows for larger time steps, though it should be chosen with care (Yagoda-Biran & Hatzor, 2016). Application of *damping* is indispensable in 3DEC not only when the aim is to find the equilibrium state of a complex system, but also to ensure numerical stability; in DDA the decision to use or not to use some kind of a damping depends on the judgement of the user.
- The memory requirements of 3DEC are lower and the calculations are, in general, faster than in DDA, particularly if the topology of the modelled system significantly varies during the process (e.g. several contacts open up or many pairs of blocks slide along each other to form new contacts with other blocks); DDA is more advantageous than 3DEC in situations when the topology does not change much and the simulated phenomenon is quasi-static or, at least, the system remains close to equilibrium.

APPLICATIONS

DDA has been applied in the analysis of scientific issues as well as in practical engineering problems. A few characteristic examples are given below.

Bicanic and Stirling (2001) applied DDA to approximately check the minimal wall thickness of a semicircular arch predicted by Heyman. They built an arch model with a 8% lower thickness and another arch with 6% higher thickness than the analytical prediction, and found that with high frictional resistance and cohesion in the joints and zero tensile strength, the first arch collapsed and the second arch was stable. Then they analysed how the arch behaviour changed when applying tensional resistance, zero friction or zero cohesion. They also analysed the effect of applying a crushing criterion for large compressive stresses at the corners of the blocks.

Rizzi et al (2014) analysed a similar problem. Correcting a few errors in Heyman's analysis, different improved solutions were derived or re-derived, and then DDA-simulated experiments on discretized arches were performed.

A damaged arch of a 4th-century church in Mamshit was modelled by Kamai and Hatzor (2005). They simulated different dynamic effects which had possibly led to the damage that could be seen on the arch, and also investigated the necessary damping to be applied in the numerical model. They could

find an acceptable approximative peak ground acceleration that could cause the damages. A similar back-analysis was done on the model of a Roman tower in Avdat. Though both structures could possibly experience more than one earthquake during their history and the separation of these different earthquakes was far beyond the scope of the analysis, the structural damages of the two structures were successfully reproduced by DDA. Similar back analysis was reported by Kamai and Hatzor (2008) on an asymmetric arch in the Nimrod fortress.

Yagoda-Biran and Hatzor (2010) dealt with on the collapsed columns of the Byzantine cathedral in Susita. In order to verify the contact normal stiffness in the model, rocking motion of a free-standing column was repeatedly simulated using different values for the stiffness, until good agreement was reached between simulation and the analytical solution. Then the verified model was applied for finding the required the peak ground acceleration for overturning a typical Susita column.

Jiang et al (2014) analysed a centuries-old masonry seawall at Qiantang River. (Besides its historical and cultural value, this seawall plays a basic role in protecting a prosperous area in southeast China against floods and tides.) The authors applied a displacement-dependent cohesional shear strength model in the contacts, and the soil domain of the model was discretized in such a way that the joints would allow for sliding along the potential failure surfaces predicted by the most important soil mechanics theories. In addition to gravity, effects like mortar or pile corrosion, soil strength degradation etc. were simulated and failure modes with corresponding safety factors were determined.

These examples demonstrate the high potential of DDA not only as a scientific research tool but also as a technique that can assist the solution of practical structural mechanical problems related to masonry structures. Though DDA simulations of realistic problems can still be rather time-consuming today, the development of hardware, intelligent parallelization, and powerful equation solvers may enable the application of DDA in the everyday engineering practice.

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KEY TERMS AND DEFINITIONS

Generalized Displacement Vector in DDA: A hypervector that consists of as many blocks as the number of the elements. The p-th block contains the translation vector of the reference point of element p, the rotation vector about the reference point, and the components of the strain tensor of p.

Implicit Method: A time integration method is implicit if the approximated numerical solution of the differential equation of the analyzed initial value problem is calculated at the end of a time interval from the numerical solution at the beginning of the interval in such a way that the differential equation is satisfied at the end of the time interval according to any required exactness.

Open-Close Iterations: The contacts in a DDA model may be in three different states: locked, sliding or open. After calculating the characteristics of the system at the end of the analyzed time interval, the assumed contact states may turn out to be inconsistent with the contact forces or element positions. In this case DDA modifies the contact states, and repeats the calculation of the time interval with the new contact states. This procedure is called open-close iterations.

Time Integration: A procedure to quantitatively approximate the solution of an initial value problem described by a differential equation. The time integration provides quantitative approximations of the state variables of the modelled system at a series of separate time instants.

Uniform Strain Element: An element type (originally applied in DDA model) in which the strain tensor is the same in every point.