7. THE DDA MODEL

7.1 Introduction

„DDA” stands for „Discontinuous Deformation Analysis”, suggesting that the displacement fields show abrupt changes on the element boundaries in the model. (This is very different from a finite element model where continuity conditions have to hold at the common nodes of neighbouring elements.)

The first DDA model was published by Gen-Hua Shi, the doctorial student of Berkeley, in his PhD dissertation in 1988. This was a two-dimensional model, with uniform-strai polygonal elements and the undeformable contacts carried normal and tangentional forces. Shi suggested his model for the analysis of fractured rocks. In the next decades several versions of DDA were published: two-dimensional and three-dimensional models were born, different element shapes were applied (e.g. spheres, Zhao et al, 2000), higher-order displacement fields were used (MacLaughlin, 1997) etc. The method has been widely applied for the analysis of masonry structures like arches or stone bridges.

The introduction below focuses on the 3D model of Shi, and based on this, the interested reader can easily understand the background of other DDA models in the literature.

7.2 The elements

The elements in Shi’s model have arbitrary polyhedral shapes. Each element has a reference point, which is the same as the centre of gravity of the element. Denote the position of the reference point of element *p* by (x^p, y^p, z^p).

The element can translate and rotate as a rigid body (6 degrees of freedom); but in addition to that, it is able for uniform deformations characterized by the usual small strain tensor. Together with the 6 characteristics of the strain, the element has altogether 12 degrees of freedom, and the 12 unknown kinematic characteristics are summarized into the generalized displacement vector of the element:

$$\mathbf{u}^p = \begin{bmatrix} u_x^p \\ u_y^p \\ u_z^p \\ \varphi_x^p \\ \varphi_y^p \\ \varphi_z^p \\ \varepsilon_x^p \\ \varepsilon_y^p \\ \varepsilon_z^p \\ \gamma_{yz}^p \\ \gamma_{zx}^p \\ \gamma_{xy}^p \end{bmatrix}$$

The translations of an arbitrary (x, y, z) point on the element can uniquely be determined with the help of the $\mathbf{u}^p$ generalized displacement vector of the element:

$$\begin{bmatrix} u_x(x,y,z) \\ u_y(x,y,z) \\ u_z(x,y,z) \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & (z-z^p) & -(y-y^p) & (x-x^p) & 0 & 0 & 0 & \frac{(z-z^p)}{2} & \frac{(y-y^p)}{2} \\ 0 & 1 & 0 & -(z-z^p) & 0 & (x-x^p) & 0 & (y-y^p) & 0 & \frac{(z-z^p)}{2} & 0 & \frac{(x-x^p)}{2} \\ 0 & 0 & 1 & (y-y^p) & -(x-x^p) & 0 & 0 & 0 & (z-z^p) & \frac{(y-y^p)}{2} & \frac{(x-x^p)}{2} & 0 \end{bmatrix} \begin{bmatrix} u_x^p \\ u_y^p \\ u_z^p \\ \varphi_x^p \\ \varphi_y^p \\ \varphi_z^p \\ \varepsilon_x^p \\ \varepsilon_y^p \\ \varepsilon_z^p \\ \gamma_{yz}^p \\ \gamma_{zx}^p \\ \gamma_{xy}^p \end{bmatrix} =$$

$$= \mathbf{T}(x,y,z)\,\mathbf{u}^p$$

The role of the **T** matrix is the same as that of the transition matrix in FEM; the difference is that in DDA it takes into account the deformations of the element too.

The forces acting on the elements are either external (like gravity or velocity-dependent drag force), or contact forces from the neighbouring elements. Consider element p, and all the forces acting on it, either external or contact forces. If these are reduced for the reference point of the element, the components are 6 scalar static characteristics. In addition, corresponding to the uniform strain, a uniform stress field is assigned to element p which has 6 more scalar characteristics (in the original model of Shi a linearly elastic stress-strain relation was applied, but more complicated constitutive are no problem to incorporate). So, altogether 12 components form the $\mathbf{f}^p$ generalized reduced force vector of element p:

$$\mathbf{f}^p = \begin{bmatrix} f_x^p \\ f_y^p \\ f_z^p \\ m_x^p \\ m_y^p \\ m_z^p \\ V^p\sigma_x^p \\ V^p\sigma_y^p \\ V^p\sigma_z^p \\ V^p\tau_{yz}^p \\ V^p\tau_{zx}^p \\ V^p\tau_{xy}^p \end{bmatrix}$$

It is important to emphasize that the uniform stress field in the element does not hold a direct relationship with the contact forces: the well-known static boundary condition **σ n** = **q** is not valid for the DDA elements and their contact forces.

Note that if an element performing an incremental displacement $d\mathbf{u}^p$ is acted upon a generalized force $\mathbf{f}^p$, the scalar product $(d\mathbf{u}^p)^T\,\mathbf{f}^p$ means work increment. (This was the reason to multiply the lower 6 components of $\mathbf{f}^p$ with the V^p volume of the element.)

Consider now the force $\mathbf{F}^c$ acting on the analysed element in point *c*. This is reduced to the reference point with the help of the transpose of the matrix **T**, so the term $(\mathbf{T}^T\,\mathbf{F}^c)$ is added to other forces when compiling the reduced force $\mathbf{f}^p$:

$$\mathbf{T}^T\mathbf{F}^c = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \\ 0 & -(z-z^p) & (y-y^p) \\ (z-z^p) & 0 & -(x-x^p) \\ -(y-y^p) & (x-x^p) & 0 \\ (x-x^p) & 0 & 0 \\ 0 & (y-y^p) & 0 \\ 0 & 0 & (z-z^p) \\ 0 & \dfrac{(z-z^p)}{2} & \dfrac{(y-y^p)}{2} \\ \dfrac{(z-z^p)}{2} & 0 & \dfrac{(x-x^p)}{2} \\ \dfrac{(y-y^p)}{2} & \dfrac{(x-x^p)}{2} & 0 \end{bmatrix} \begin{bmatrix} F_x^c \\ F_y^c \\ F_z^c \end{bmatrix}$$

Irrelevantly of whether a contact force $\mathbf{F}^c$ is considered or any other concentrated external effect, the model of Shi gives the same simplified relation between forces and stresses as seen in the last six rows of the above relation. (Other DDA models with higher order strain fields work in the same manner, applying higher order stress fields, which means more kinematic and static variables belonging to the elements.)

In the practical analysis of fractured rocky soils the initial self-stress must be taken into consideration for the reliable modelling. This is done with the lower six components of $\mathbf{f}^p$.

7.3 The contacts

When two elements move into a position where they overlap each other, a contact is formed. Overlapping is resisted by penalty functions in the DDA method: a compressional contact force occurs whose magnitude depends linearly on the depth of the overlap. It is practically equivalent to an elastic, no-tension contact model. Tangential forces are treated in the same way, and the Coulomb-friction gives a limit to their magnitude.

7.4 The equations of motion

Every element has 12 scalar equations in its generalized equations of motion. The first 3 equations set the link between translational accelerations of the element and the first 3 components in the reduced force vector. In the second 3 equations the rotational accelerations and the three moment components are related. The last 6 equations (relations between stresses and time derivatives of strains) were derived by Shi from the principle of minimum potential energy; their mechanical meaning is not clarified in the literature.

To go into the details, define the generalized displacement vector and reduced force vector for the system containing N elements in the usual way:

$$\mathbf{u}(t)=\begin{bmatrix}\mathbf{u}^1(t)\\ \mathbf{u}^2(t)\\ \vdots\\ \mathbf{u}^N(t)\end{bmatrix} \quad ; \qquad \mathbf{f}(t)=\begin{bmatrix}\mathbf{f}^1(t,u(t),v(t))\\ \mathbf{f}^2(t,u(t),v(t))\\ \vdots\\ \mathbf{f}^N(t,u(t),v(t))\end{bmatrix} \; .$$

The generalized velocities and accelerations are

$$\mathbf{v}(t)=\frac{d\mathbf{u}(t)}{dt}, \quad \mathbf{a}(t)=\frac{d^2\mathbf{u}(t)}{dt^2}$$

(note that these vectors contain the time derivatives of the strains of the elements too). The equations of motion has the following form:

$$\mathbf{M}\cdot\mathbf{a}(t)=\mathbf{f}(t,\mathbf{u}(t),\mathbf{v}(t)) \; .$$

The $\mathbf{M}$ matrix of inertia has a block-diagonal structure:

$$\mathbf{M}=\begin{bmatrix}\mathbf{M}^1 & & & \\ & \mathbf{M}^2 & & \\ & & \ddots & \\ & & & \mathbf{M}^N\end{bmatrix} ,$$

where the $\mathbf{M}^p$ block belonging to the p-th element can be expressed from the $\mu(x, y, z)$ density function of the element, with the help of matrix $\mathbf{T}$ which was seen above:

$$\mathbf{M}^p=\int_{V^p}\mathbf{T}^T(x,y,z)\cdot\mathbf{T}(x,y,z)\cdot\mu(x,y,z)\,dV$$

(the details of the derivations can be found in Shi, 2001).

7.5 Time integration: Analysis of a single time step

Consider the tie interval (t_i, t_{i+1}). At its beginning, at t_i, the state of the system is known: the vectors $\mathbf{u}_i$, $\mathbf{v}_i$, $\mathbf{f}(t_i, \mathbf{u}_i, \mathbf{v}_i)$ are known, and they satisfy the equations of motion. The aim of the calculations is to ensure that the equations of motion

$$\mathbf{0}=\mathbf{f}(t,\mathbf{u}(t),\mathbf{v}(t))-\mathbf{M}\cdot\mathbf{a}(t)$$

would be satisfied also at t_{i+1}:

$$\mathbf{0}=\mathbf{f}(t_{i+1},\mathbf{u}_{i+1},\mathbf{v}_{i+1})-\mathbf{M}_{i+1}\cdot\mathbf{a}_{i+1}$$

So, the aim is to find that $\Delta\mathbf{u}_{i+1} = \mathbf{u}_{i+1} - \mathbf{u}_i$ generalized displacement increment which takes the system to the end of the timestep into a state where the equations of motion are satisfied.

In addition to $\Delta\mathbf{u}_{i+1}$ the unknown $\mathbf{v}_{i+1}$ and $\mathbf{a}_{i+1}$ are also contained in the equations of motion. Approximate them in terms of $\Delta\mathbf{u}_{i+1}$ in the following way:

$$\mathbf{a}_{i+1} = \frac{1}{\Delta t^2 / 2}\left(\Delta\mathbf{u}_{i+1} - \Delta t \cdot \mathbf{v}_i\right)$$
$$\mathbf{v}_{i+1} = \mathbf{v}_i + \Delta t \cdot \mathbf{a}_{i+1}$$

The mechanical meaning of these approximations is that the acceleration $\mathbf{a}_{i+1}$ is valid in the whole (t_i, t_{i+1}) interval, independently of the accelerations at the end of the previous time interval, i.e. at t_i. Mathematically these approximations correspond to the Newmark β-method with $\beta = 1/2$; $\gamma = 1$.

Inserting them into the equations of motion, $\Delta\mathbf{u}_{i+1}$ remains the only unknown. Let us give a new notation to the right side:

$$\mathbf{f}(t_{i+1}, \mathbf{u}_{i+1}, \mathbf{v}_{i+1}) - \mathbf{M}_{i+1} \cdot \mathbf{a}_{i+1} := \mathbf{r}(t_{i+1}, \Delta\mathbf{u}_{i+1})$$

which means that the aim is to find that$\Delta\mathbf{u}_{i+1}$ which makes **r** zero at t_{i+1}.

In DDA the unknown $\Delta\mathbf{u}_{i+1}$ is calculated with the help of the Newton-Raphson iteration method where the Jacobian matrix of $\mathbf{r}(t, \Delta\mathbf{u})$ by $\Delta\mathbf{u}$ is needed:

$$\mathcal{K}(t, \Delta\mathbf{u}) = \frac{d\mathbf{r}(t, \Delta\mathbf{u})}{d\Delta\mathbf{u}}$$

This matrix is the sum of the usual stiffness matrix based on the contact stiffnesses, and of other terms due to the inertia and the stiffnesses of the material of the elements. In his publications Shi gave these derivations in detail (e.g. Shi, 2001), so the calculation of the Jacobian is straightforward. Hence, the residual for a given t and $\Delta\mathbf{u}$ can uniquely be calculated – the interested reader should consult the literature for the exact (rather complicated) details.

Focus now on how to determine the unknown $\Delta\mathbf{u}_{i+1}$. Let its first approximation be a zero vector, and for the beginning of the calculations apply the already known position, velocity and internal forces of the system valid at t_i:

$$\Delta\mathbf{u}_{i+1}^{(0)} := \mathbf{0} \ .$$

If $\mathbf{r}(t_{i+1}, \Delta\mathbf{u}_{i+1}^{(0)}) = \mathbf{0}$ were valid, the calculation would be ready: the equations of motion would be satisfied. However, typically this is not the case, so a better approximation can be produced as

$$\Delta\mathbf{u}_{i+1}^{(1)} := \Delta\mathbf{u}_{i+1}^{(0)} - \mathcal{K}(t_{i+1}, \Delta\mathbf{u}_{i+1}^{(0)})^{-1} \cdot \mathbf{r}(t_{i+1}, \Delta\mathbf{u}_{i+1}^{(0)}) \ .$$

Check now whether $\mathbf{r}(t_{i+1}, \Delta\mathbf{u}_{i+1}^{(1)})$ is sufficiently small. The iteration can be terminated if its norm is below a pre-defined threshold value; otherwise, a next approximation has to be prepared, according to the general formula:

$$\Delta\mathbf{u}_{i+1}^{(k+1)} := \Delta\mathbf{u}_{i+1}^{(k)} - \mathcal{K}(t_{i+1}, \Delta\mathbf{u}_{i+1}^{(k)})^{-1} \cdot \mathbf{r}(t_{i+1}, \Delta\mathbf{u}_{i+1}^{(k)}) \ .$$

The iteration is continued until $\left|\mathbf{r}(t_{i+1}, \Delta\mathbf{u}_{i+1}^{(k+1)})\right|$ decreases below the threshold. (In some methods the norm $\left|\Delta\mathbf{u}_{i+1}^{(k+1)} - \Delta\mathbf{u}_{i+1}^{(k)}\right|$ is also checked, which is particularly useful in the case of nearly-singular $\mathcal{K}$.)

After finding $\Delta\mathbf{u}_{i+1}$ with sufficient accuracy, the next task is to check whether the topology of the system remains unchanged, or the displacements led to new and cancelled contacts. If an overlap is found between two elements not in contact at the beginning of the timestep, DDA returns to the beginning of the timestep, puts a compressional spring between the two elements, and the calculation of the timestep is repeated. Similarly, if a contact is lost because of the calculated displacements, the contact is removed and the calculation of the timestep is repeated without that contact. If the friction limit is exceeded by a tangential force in a contact, then the timestep is repeated by keeping the shear force at the friction limit, and setting the shear stiffness to zero.

Of course, during the re-calculation of a timestep, other contacts may be modified or born. In this case the calculation is repeated again and again, until no change is experienced. (The convergence of the method has not been proven yet.)

After finding the new state which satisfies the equations of motion as well as all conditions related to the contact behaviour, a next timestep can be analysed.

Remarks:

1. In the literature the equations of motion are often written in this form:

$$\mathbf{Ma}(t) + \mathbf{Cv}(t) + \mathbf{K}\Delta\mathbf{u}(t) = \mathbf{f}(t)$$

Here matrix **C** serves for the calculation of drag forces, and **K** the classic stiffness matrix, which is ***not the same*** as matrix $\mathcal{K}$ seen above.

2. Shi derived the compilation of $\mathcal{K}$ and **r** on the basis of potential energy minimalisation, taking into consideration among the terms the $-\mathbf{Ma}(t)$ component in the form of D'Alembert-forces. (In his publications the matrix denoted by **K** is the same as the matrix $\mathcal{K}$ above in this section; but his notations on **r** and other quantities here are very different from those applied in this section.)

7.6 Applications

Several nice engineering applications have been published on the DDA methods. Some of them are:

- Three Gorges Dam in China (Zhu et al, 1999);

– plan of reinforcement of Masada mountain in Israel, Hatzor et al, 2002;

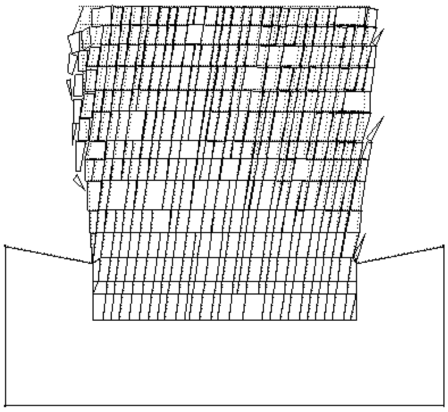

Figure 1.
Masada mountain and its two-dimensional DDA model (Hatzor et al, 2002)

– Mechanical analysis of a displaced masonry arch, Kamai et al (2005):

Figure 2.
The displaced block in the arch in Kamai et al, 2005

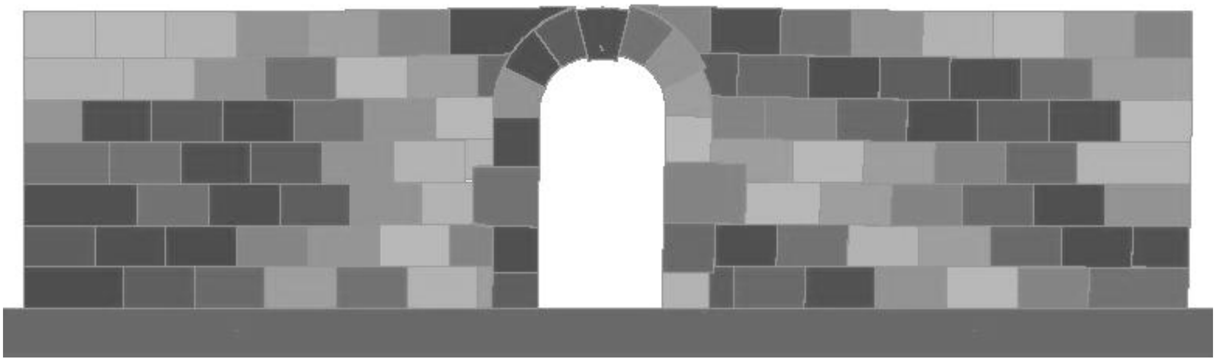

Figure 3.
DDA model of the neighbourhood of the analysed arch in Kamai et al, 2005

- The Gjovick stadium in Norway, already mentioned in Section 5 about UDEC (Scheldt et al, 2002);
- Yerba Buena tunnel in San Francisco (Law és Lam, 2003).

Questions

7.1. Explain the quantities in the generalized displacement vector and in the generalized reduced force vector in DDA!

7.2. How is the time integration done in a timestep in DDA?