8. THE CONTACT DYNAMICS METHOD

8.1 Introduction

The Contact Dynamics („CD") method was introduced at the beginning of the 1990ies by M. Jean and J.J. Moreau (Jean és Moreau, 1992; Jean, 1999). The method turned out to be extremely fast when simulating granular flows, rapid avalanches, segregation, vibration problems of granular materials etc.

Most discrete element methods take into consideration the deformability of the elements e.g. by concentrating it into the contacts, like in the case of BALL-type models, or by applying an internal finite element mesh like in UDEC. The calculation of the contact forces between the elements are based on the stiffness characteristics. The logic of CD is different. The elements are considered to be perfectly rigid, and the contact forces are not related to any stiffness data. The contact forces should ensure the equilibrium of the elements, and in addition, they must not violate requirements like e.g. Coulomb limit for friction, or no tension in cohesionless contacts, but their calculation does not apply any constitutive relations. For statically highly indeterminate systems like e.g. granular assemblies in general, there exist several valid force systems satisfying the equilibrium conditions; CD produces randomly one of them, and several equally valid solutions can be received if the problem is calculated again and again with the Contact Dynamics Method. No wonder that its application is not very widespread in the engineering practice; on the other hand, since the method is computationally very efficient for the simulation of dynamic problems, it is rather popular in the granular physics literature.

The original papers on CD were rather difficult to decipher. The paper of Unger and Kertész (2003) was very different from them in this respect: the authors gave a clear, code-writer-oriented introduction to the main line of thought of the method, giving a significant help this way for those who wanted to write their own code and also for those who just wanted to understand how the method worked which they were applying in their researches. The introduction below is also based on this publication.

8.2 The elements and the contacts

Unlike in other methods where the basic unit of the analysis is the element or a part of the element around the node, in CD the basic unit is the ***pair of neighbouring elements***. The aim of the calculations is to iteratively find the suitable contact forces in all pairs in the system.

The elements in the original version of the CD method were circular and perfectly rigid. Later several publications appeared in the literature on CD models with non-circular and even on deformable elements; but for simplicity, the introduction below will focus only on perfectly rigid, perfectly spherical three-dimensional elements. These elements have six degrees-of-freedom (3 translations and 3 rotations). A reference point is defined for each element coinciding with its centre of gravity.

Focus now on a selected pair of two elements close to (perhaps in contact with) each other. We shall collect the state variables and the mechanical characteristics of this pair.

Let g^{pq} denote the distance between the elements p and q. If this distance equals to zero, the two elements get into a point-like contact (denote it by c), and a concentrated contact force can be transmitted between the elements. Unlike in most discrete element techniques, now the contact and the elements do not deform, so contacting elements must move in such a way that there would be no overlap between them: g^{pq} cannot become negative.

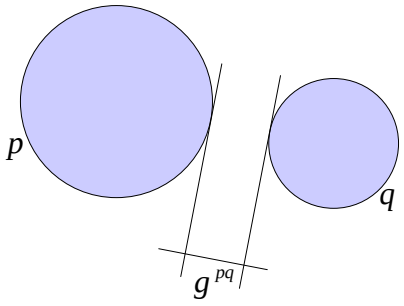

Figure 4.
The distance g^{pq} between the elements p and q

The force acting on p by q is $\mathbf{f}^{pc}$, and the force expressed on q by p is $\mathbf{f}^{qc}$:

$$\mathbf{f}^{pc} = \begin{bmatrix} F_x^{pc}(t) \\ F_y^{pc}(t) \\ F_z^{pc}(t) \end{bmatrix} \text{ and } \mathbf{f}^{qc} = \begin{bmatrix} -F_x^{pc}(t) \\ -F_y^{pc}(t) \\ -F_z^{pc}(t) \end{bmatrix} ,$$

and these can be reduced to the reference points of p and q with the help of the $\mathbf{B}^{pc}(t)$ and $\mathbf{B}^{pc}(t)$ transition matrices, respectively:

$$\mathbf{B}^{pc}(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -r_z^{pc} & r_y^{pc} \\ r_z^{pc} & 0 & -r_x^{pc} \\ -r_y^{pc} & r_x^{pc} & 0 \end{bmatrix} \quad ; \quad \mathbf{B}^{qc}(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -r_z^{qc} & r_y^{qc} \\ r_z^{qc} & 0 & -r_x^{qc} \\ -r_y^{qc} & r_x^{qc} & 0 \end{bmatrix} .$$

The vectors $\mathbf{r}^{pc}$ and $\mathbf{r}^{qc}$ point from the reference points to the contact. The reduced forces are then:

$$\mathbf{f}_{red}^{pc}(t) = \mathbf{B}^{pc}(t) \cdot \mathbf{f}^{pc}(t) \quad ; \quad \mathbf{f}_{red}^{qc}(t) = \mathbf{B}^{qc}(t) \cdot \mathbf{f}^{qc}(t) \quad .$$

Summarize these two forces into the 12-scalar ***reduced contact force vector*** of the pair::

$$\mathbf{f}^{pq}(t) = \begin{bmatrix} \mathbf{f}_{red}^{pc}(t) \\ \mathbf{f}_{red}^{qc}(t) \end{bmatrix} .$$

This vector will take part in the equations of motion of the pair.

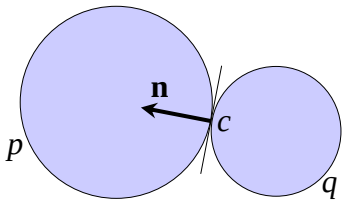

Figure 5.
The normal vector of the pq contact

Assign an (**n**, **t**, **w**) local coordinate frame to the contact: **n** is the common unit normal of the contact plane, pointing towards element *p* (see Figure 5); **t** and **w** are an arbitrary pair of orthogonal unit vectors in the contact plane.

The vector $\mathbf{v}^{pq}$ denotes the velocity vector of the (*p*, *q*) pair:

$$\mathbf{v}^{pq}(t) = \begin{bmatrix} \mathbf{v}^{p}(t) \\ \mathbf{v}^{q}(t) \end{bmatrix} ,$$

which can be expressed as the time derivatives of the displacements:

$$\mathbf{v}^{p}(t) = \begin{bmatrix} \frac{du_x^p(t)}{dt} \\ \frac{du_y^p(t)}{dt} \\ \frac{du_z^p(t)}{dt} \\ \frac{d\varphi_x^p(t)}{dt} \\ \frac{d\varphi_y^p(t)}{dt} \\ \frac{d\varphi_z^p(t)}{dt} \end{bmatrix} ; \quad \mathbf{v}^{q}(t) = \begin{bmatrix} \frac{du_x^q(t)}{dt} \\ \frac{du_y^q(t)}{dt} \\ \frac{du_z^q(t)}{dt} \\ \frac{d\varphi_x^q(t)}{dt} \\ \frac{d\varphi_y^q(t)}{dt} \\ \frac{d\varphi_z^q(t)}{dt} \end{bmatrix} .$$

Consider now those two material points, *pc* and *qc*, which form the contact. Their velocities, $\mathbf{v}^{pc}$ és $\mathbf{v}^{qc}$, can be expressed again with the help of the $\mathbf{B}^{pc}(t)$ and $\mathbf{B}^{pc}(t)$ transition matrices:

$$\begin{aligned} \mathbf{v}^{pc}(t) &= \mathbf{B}^{pcT}(t) \cdot \mathbf{v}^{p}(t) \\ \mathbf{v}^{qc}(t) &= \mathbf{B}^{qcT}(t) \cdot \mathbf{v}^{q}(t) \end{aligned} .$$

The relative velocity of the contact is the difference of the velocities of the two material points *pc* and *qc*:

$$\boldsymbol{\mu}^{pq}(t)=\begin{bmatrix}\mu_x^{pq}(t)\\ \mu_y^{pq}(t)\\ \mu_z^{pq}(t)\end{bmatrix}:=\mathbf{v}^{pc}(t)-\mathbf{v}^{qc}(t)$$

which yields

$$\boldsymbol{\mu}^{pq}(t)=\mathbf{v}^{pc}(t)-\mathbf{v}^{qc}(t)=\left(\mathbf{B}^{pcT}(t)\cdot\mathbf{v}^{p}(t)\right)-\left(\mathbf{B}^{qcT}(t)\cdot\mathbf{v}^{q}(t)\right)\ .$$

This vector shows the relative translation of point *pc* with respect to point *qc*, so if the two points are already in contact, the n-component of this vector must be positive to exclude overlapping.

Contacts in the CD method are modelled in the following way:

(1) If the two elements are in contact, i.e. if $g^{pq} = 0$, then a contact force can be transmitted. If $g^{pq} > 0$, then there is no contact, and no contact force exist in the pair. The case $g^{pq} < 0$ is not possible.
(2) The normal component of the contact force, N^{pq}, can only be negative, i.e. compressional, but otherwise its magnitude ($N^{pq}=\mathbf{f}^{pcT}\cdot(-\mathbf{n})$ vagy $N^{pq}=\mathbf{f}^{qcT}\cdot\mathbf{n}$) is arbitrary. The (N^{pq}, g^{pq}) relation is shown in Figure 6.:

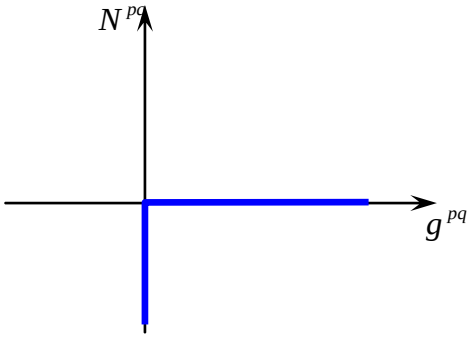

Figure 6.
Normal force versus gap width in CD

(3) The magnitude of the tangential $\mathbf{T}^{pq}$ (which has a *t* and a *w* component) is limited by the Coulomb friction law:

$$\left|\mathbf{T}^{pq}\right|\le-\boldsymbol{\nu}\cdot N^{pq}$$

where ν is the friction coefficient. Figure 7. illustrates this limitation: the vector of the tangential force must point from the origin either to inside the cone (non-sliding contact) or just to the surface of the cone (sliding contact).

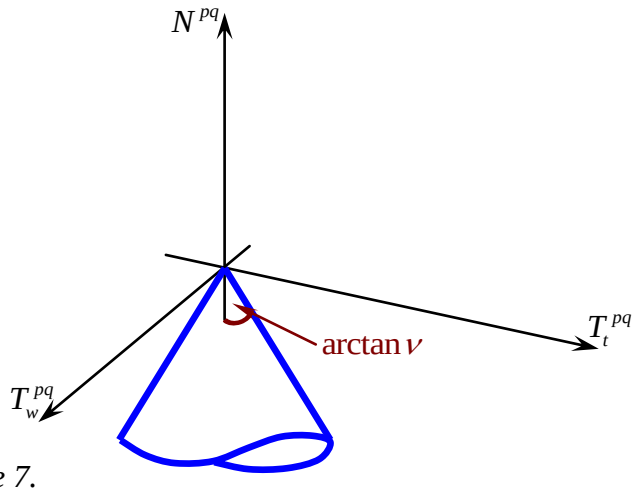

Figure 7.
Columb limit for the tangential force

As long as $\mathbf{T}^{pq}$ is below the friction limit, the tangential component of the relative translation must be zero (elastic deformation not possible). When reaching the friction limit, the ontact starts to slide: the $\boldsymbol{\mu}^{pq}(t)$ relative velocity has to be just opposite to the direction of the $\mathbf{T}^{pq}$ force (the tangential relative translation must not have a component perpendicularly to the frictional force reaching the limit), and the magnitude of $\mathbf{T}^{pq}$ is equal to $-\nu\cdot\mathbf{N}^{pq}$. The magnitude of the relative translation is not limited in the contact model, this can be determined from kinematical considerations.

8.3 The equations of motion

CD is a time-stepping method. Its fundamental unknowns are the time-dependent positions and velocities of the elements; however, they are not compiled into hypervectors like in other discrete element methods, because they are analysed separately pair by pair.

Assume that at t_i the state of the system is known: as far as numerically possible, the positions and velocities of the elements are given: $\mathbf{u}^p(t_i)\cong\mathbf{u}_i^p$; $\mathbf{v}^p(t_i)\cong\mathbf{v}_i^p$, and the external forces acting on the elements ($\mathbf{f}_i^{p,ext}$) and contact forces for all c ($\mathbf{f}_i^{pc}$) are also known. The external forces are reduced to the reference points; the contact forces act in the point-like contacts. The time-dependence of the external forces is also known (e.g. that the gravitational force is constant), so the external forces are given also in $t_{i+1}=t_i+\Delta t$ ($\mathbf{f}_{i+1}^{p,ext}$). From these data the state of the system at t_{i+1} (contact forces, the positions and the velocities of the elements) is searched for.

CD applies the ***implicit*** version of the Euler method for this. The basic step for the p and q pair can be written as:

$$\begin{bmatrix}\mathbf{v}_{i+1}^p\\ \mathbf{v}_{i+1}^q\end{bmatrix}:=\begin{bmatrix}\mathbf{v}_i^p\\ \mathbf{v}_i^q\end{bmatrix}+\Delta t\cdot\begin{bmatrix}(\mathbf{M}^p)^{-1} & \\ & (\mathbf{M}^q)^{-1}\end{bmatrix}\cdot\begin{bmatrix}\mathbf{f}_{i+1}^p\\ \mathbf{f}_{i+1}^q\end{bmatrix};$$

$$\begin{bmatrix}\mathbf{u}_{i+1}^p\\ \mathbf{u}_{i+1}^q\end{bmatrix}:=\begin{bmatrix}\mathbf{u}_i^p\\ \mathbf{u}_i^q\end{bmatrix}+\Delta t\cdot\begin{bmatrix}\mathbf{v}_{i+1}^p\\ \mathbf{v}_{i+1}^q\end{bmatrix}.$$

Here $\mathbf{f}_{i+1}^{p}$ and $\mathbf{f}_{i+1}^{q}$ denote the resultants of the external and all contact forces acting on p and q respectively, being reduced to the reference points. (Note that according to the implicit scheme, the velocities and accelerations belonging to the end of the time interval are considered to be valid along the whole interval.) These reduced forces are

$$\begin{bmatrix} \mathbf{f}_{i+1}^{p} \\ \mathbf{f}_{i+1}^{q} \end{bmatrix} := \begin{bmatrix} \mathbf{f}_{i+1}^{p,külső} + \sum_{(pk)} \mathbf{B}^{pk} \cdot \mathbf{f}_{i+1}^{pk} \\ \mathbf{f}_{i+1}^{q,külső} + \sum_{(qk)} \mathbf{B}^{qk} \cdot \mathbf{f}_{i+1}^{qk} \end{bmatrix} .$$

Summation over index pk runs along all contacts of element p , including the just analysed contact c as well. Similarly, index qk runs along all contacts of q.

The transition matrices are assumed to be constant during the (t_i, t_{i+1}) time interval, and equal to their values at t_i. Indeed, if the displacement increments are small during the timestep, the modification of the vectors pointing from the reference points to the contacts is negligible.

Collect the mass and rotational inertia of the elements into the matrices $\mathbf{M}^p$ and $\mathbf{M}^q$, which have the following form in the case of spherical elements:

$$\mathbf{M}^p = \begin{bmatrix} m^p & & & & & & \\ & m^p & & & & & \\ & & m^p & & & & \\ & & & I^p & & & \\ & & & & I^p & & \\ & & & & & I^p & \\ & & & & & & I^p \end{bmatrix} .$$

(Note that because of the spherical symmetry, these matrices are constant in time.)

In order to determine the velocity of the pair ($\mathbf{v}_{i+1}^{pq}$), the resultants $\mathbf{f}_{i+1}^{p}$ and $\mathbf{f}_{i+1}^{q}$ should be known. So, in addition to the external forces acting at t_{i+1}, the contact forces should also be known at the end of the timestep. The Contact Dynamics models search for these contact forces with the help of an ***iterative solver*** (which has to be performed over and over again at every timestep, as the contact forces change with time).

8.4 The iterative solver

The solver sweeps along all pairs of neighbouring or nearly contacting elements. When considering a given pair, an approximation is given (based on the equations of motion of the pair), so that the conditions assumed on the mechanical behaviour would be satisfied: no overlap; Coulomb-friction etc. Then a next pair is considered. After all pairs were swept over, the solver starts the pairs from the beginning, and it is repeated over and over again, until the next approximations are already sufficiently close to the previous ones. It means that the contact forces belonging to t_{i+1} have been found, and the next time step can follow.

The approximation of the contact force in the pair (p, q) is based on the equations of motion of that pair. Before turning onto the details, a few notations have to be introduced:

For element p, reduce to the reference point all those forces (external and contact forces) acting at t_{i+1}, ***except from*** the force expressed by element q through contact c:

$$\mathbf{f}_{red,i+1}^{p,no\_c} := \mathbf{f}_{i+1}^{p,ext} + \sum_{pk \neq pc} \mathbf{f}_{red,i+1}^{pk} \quad .$$

The $\mathbf{f}_{i+1}^{pk}$ contact force is only an actual approximation of the force indeed acting in contact pk at t_{i+1}; it receives new and new values during the iterations. (At the beginning of the analysis of the time step the contact forces are approximated to be the same as their final, just determined values at the end of the previous time step, which is the same as the beginning of the just analysed timestep.)

Similarly, reduce all the forces acting at t_{i+1} on q – except from that force acting in qc – to the reference point of q:

$$\mathbf{f}_{red,i+1}^{q,no\_c} := \mathbf{f}_{i+1}^{q,ext} + \sum_{qk \neq qc} \mathbf{f}_{red,i+1}^{qk} \quad ,$$

and collect the two vectors into a hypervector:

$$\mathbf{f}_{red,i+1}^{pq,no\_c} := \begin{bmatrix} \mathbf{f}_{red,i+1}^{p,no\_c} \\ \mathbf{f}_{red,i+1}^{q,no\_c} \end{bmatrix} \quad .$$

Summarize the two transition matrices belonging to c into a hypermatrix:

$$\mathbf{B}^{pq} := \begin{bmatrix} \mathbf{B}^{pc} \\ -\mathbf{B}^{qc} \end{bmatrix}$$

and the matrices of inertia of p and q into a block-diagonal matrix, whose inverse is:

$$\left(\mathbf{M}^{pq}\right)^{-1} := \begin{bmatrix} \left(\mathbf{M}^{p}\right)^{-1} & \mathbf{0} \\ \mathbf{0} & \left(\mathbf{M}^{q}\right)^{-1} \end{bmatrix} \quad .$$

Later the following two matrices will also be necessary:

$$\left(\mathbf{M}^{pq}\right)^{-1} := \mathbf{B}^{pqT} \left(\mathbf{M}^{pq}\right)^{-1} \mathbf{B}^{pq} \quad ,$$

and

$$\mathbf{M}^{pq} := \left(\mathbf{B}^{pqT} \left(\mathbf{M}^{pq}\right)^{-1} \mathbf{B}^{pq}\right)^{-1} \quad .$$

And now the equations of motion of the pair (p, q) can be compiled. First, the equations belonging to the end of the time interval can separately be written as:

$$\frac{1}{\Delta t} \begin{bmatrix} \mathbf{v}_{i+1}^{p} - \mathbf{v}_{i}^{p} \\ \mathbf{v}_{i+1}^{q} - \mathbf{v}_{i}^{q} \end{bmatrix} = \left(\mathbf{M}^{pq}\right)^{-1} \cdot \begin{bmatrix} \mathbf{f}_{red,i+1}^{p,no\_c} + \mathbf{f}_{red,i+1}^{pc} \\ \mathbf{f}_{red,i+1}^{q,no\_c} + \mathbf{f}_{red,i+1}^{qc} \end{bmatrix} \quad .$$

Multiply both sides by $\mathbf{B}^{pqT}$ from the left:

$$\frac{1}{\Delta t} \left[\boldsymbol{\mu}_{i+1}^{pq} - \boldsymbol{\mu}_{i}^{pq} \right] = \mathbf{B}^{pqT} \left(\mathbf{M}^{pq}\right)^{-1} \cdot \mathbf{f}_{red,i+1}^{pq,no\_c} + \\ + \mathbf{B}^{pqT} \left(\mathbf{M}^{pq}\right)^{-1} \mathbf{B}^{pq} \cdot \mathbf{f}_{i+1}^{pc}$$

(it was taken into consideration that $\mathbf{f}_{i+1}^{qc} = -\mathbf{f}_{i+1}^{pc}$). After some rearrangements:

$$\boldsymbol{\mu}_{i+1}^{pq} - \left(\boldsymbol{\mu}_{i}^{pq} + \Delta t \cdot \mathbf{B}^{pqT} \left(\mathbf{M}^{pq} \right)^{-1} \cdot \mathbf{f}_{red,i+1}^{pq,no\_c} \right) = \Delta t \cdot \left(\mathbf{M}^{pq} \right)^{-1} \mathbf{f}_{i+1}^{pc} \quad .$$

It is easy to notice that on the left side the vector in the parentheses means that relative velocity which would occur in the contact at t_{i+1} if $\mathbf{f}_{i+1}^{pc}$ is zero, i.e. if there is no force in the contact. This vector will have a special importance in the forthcoming derivation, so a special notation is given to it:

$$\boldsymbol{\mu}_{i+1}^{pq,no\_c} := \left(\boldsymbol{\mu}_{i}^{pq} + \Delta t \cdot \mathbf{B}^{pqT} \left(\mathbf{M}^{pq} \right)^{-1} \cdot \mathbf{f}_{red,i+1}^{pq,no\_c} \right) \quad .$$

The equations of motion can now be written as:

$$\boldsymbol{\mu}_{i+1}^{pq} = \boldsymbol{\mu}_{i+1}^{pq,no\_c} + \Delta t \cdot \left(\mathbf{M}^{pq} \right)^{-1} \mathbf{f}_{i+1}^{pc}$$

where $\boldsymbol{\mu}_{i+1}^{pq}$ and $\mathbf{f}_{i+1}^{pc}$ are the unknowns. So the equations of motion give the relation between the unknown contact force and the unknown relative velocity belonging to the contact. This will be the starting point of the forthcoming calculations.

Finally the normal and tangential components of the relative velocity vector of the contact will be needed:

$$\mu_{n}^{pq} = \mathbf{n}^{T} \cdot \boldsymbol{\mu}^{pq} \; ; \quad \boldsymbol{\mu}_{tw}^{pq} = \boldsymbol{\mu}^{pq} - \mu_{n}^{pq} \cdot \mathbf{n}$$

(Remember that $\boldsymbol{\mu}^{pq}$ denoted the velocity of the material point *pc* relative to the material point *qc*. Hence a positive μ_{n}^{pq} means increasing gap between the two material points.) Since the vector $\boldsymbol{\mu}_{i+1}^{pq,no\_c}$ belonging to the time instant t_{i+1} can directly be calculated from the already existing approximations of all other contact forces except from *c*, the components $\mu_{n,i+1}^{pq,no\_c}$ and $\boldsymbol{\mu}_{tw,i+1}^{pq,no\_c}$ can also be determined, while the components of the vector $\boldsymbol{\mu}_{i+1}^{pq}$ are unknowns.

The unknown $\boldsymbol{\mu}_{i+1}^{pq}$ and $\mathbf{f}_{i+1}^{pc}$ vectors are determined in three steps:

Step 1. First decide whether the two elements will be in contact at t_{i+1} : calculate how large will the gap be between them, assuming zero contact force:

$$g_{i+1}^{pq,no\_c} = g_{i}^{pq} + \mu_{n,i+1}^{pq,no\_c} \cdot \Delta t \quad .$$

A positive result means that there will be no contact at t_{i+1} , and the analysis of another pair can immediately follow. A negative result, on the other hand, means that without a contact force the elements *p* and *q* would overlap, so an $\mathbf{f}^{pc}$ contact force is needed to avoid the overlap. In this case Step 2. follows.

Step 2. The contact force should modify the velocities of the two elements in such a way that instead of overlapping, they would exactly touch each other at the end of the time step. In Step 2. the aim is to determine $\mathbf{f}_{i+1}^{pc}$ that satisfies the following two conditions:

(i) at t_{i+1} the gapwidth between *p* and *q* is exactly zero:

$$g_{i}^{pq} + \mu_{n,i+1}^{pq} \cdot \Delta t = 0$$

(ii) the contact does not slide, so the tangential component of the relative translation is zero:

$$\left| \boldsymbol{\mu}_{tw,i+1}^{pq} \right| = 0$$

To satisfy these two conditions, the relative velocity of the contact should be:

$$\boldsymbol{\mu}_{i+1}^{pq} = -\frac{1}{\Delta t} g_i^{pq} \cdot \mathbf{n}$$

(the ngative sign means that if the gapwidth was larger than zero, then *p* should get closer to *q* to touch it).

The $\mathbf{f}_{i+1}^{pc}$ has to be such a force that if continuously acting between *p* and *q* during (t_i, t_{i+1}), at t_{i+1} the relative velocity would be just equal to $\boldsymbol{\mu}_{i+1}^{pq}$. From the equations of motion, this force turns out to be equal to:

$$\mathbf{f}_{i+1}^{pc} = \frac{1}{\Delta t}\mathbf{M}^{pq} \cdot \left(-\frac{1}{\Delta t} g_i^{pq}\mathbf{n} - \boldsymbol{\mu}_{i+1}^{pq,no\_c} \right) .$$

Now the question is whether this force violates the constitutive conditions. There were two conditions on the components of the contact forces. The first one required the normal force a compression. This is automatically satisfied because of Step 1. The second one was the Coulomb-condition:

$$\left|\mathbf{T}_{i+1}^{pc}\right| \le -\boldsymbol{\nu} \cdot N_{i+1}^{pc}$$

If this holds for the calculated $\mathbf{f}_{i+1}^{pc}$, then the analysis of the (*p*, *q*) pair is ready, and a next pair can follow. However, if the tangential component exceeds the friction limit, then the calculated contact force cannot be transmitted in the contact: the contact slides, which means that the tangential component of $\boldsymbol{\mu}_{i+1}^{pq}$ is not zero, and the calculation based on zero tangential component should be corrected. This correction is done in Step 3.

Step 3. In a sliding contact the tangential force component has to satisfy the following to conditions, and – as the third condition – the equations of motion:

(i) The contact is sliding, so the magnitude of the tangential force component is equal to the Coulomb-limit:

$$\left|\mathbf{T}_{i+1}^{pc}\right| = -\boldsymbol{\nu} \cdot N_{i+1}^{pc}$$

(ii) The direction of the tangential relative velocity is just opposite to the direction of the tangential force component:

$$\frac{\mathbf{T}_{i+1}^{pc}}{\left|\mathbf{T}_{i+1}^{pc}\right|} = -\frac{\boldsymbol{\mu}_{tw,i+1}^{pq}}{\left|\boldsymbol{\mu}_{tw,i+1}^{pq}\right|}$$

(iii) the equations of motion:

$$\mathbf{f}_{i+1}^{pc} = -\frac{1}{\Delta t}\mathbf{M}^{pq} \cdot \left(\frac{1}{\Delta t} g_i^{pq}\mathbf{n} + \boldsymbol{\mu}_{i+1}^{pq,no\_c} - \boldsymbol{\mu}_{tw,i+1}^{pq} \right)$$

From these conditions the unknowns $\mathbf{f}_{i+1}^{pc}$ és $\boldsymbol{\mu}_{i+1}^{pq}$ can be calculated, and the analysis of the (*p*, *q*) pair is ready. The next pair can follow.

These calculations introduced above give an approximation for the contact force in a pair, assuming that all other contact forces are unchanged and keep their values last approximated. When turning to the next pair, the latest approximations in other pairs are applied. Proceeding from pair to pair this way, an approximation is received for the whole system of contact forces. By sweeping through the complete set of contacts over and over again, the results (at least, hopefully) get closer and closer to what should really exist at t_{i+1}. The modifications caused by the consecutive iteration cycles cause smaller and smaller modifications in the contact forces; and the iteration can be terminated as the modifications

decrease under a prescribed threshold. Now the state belonging to t_{i+1} has been found, and a new time step can be analysed.

The convergence of the method is an open question in the literature.

The order according to which the pairs are considered within an iteration step is ***random***; the only requirement is that every pair should be considered once. In the next iteration steps the ordering is different, prescribed also by a random number generator.

If the same problem is analysed twice, by starting the random number generator from two different places, the two resulting contact force systems will be different. This is evident for an engineer, particularly if equilibrium is searched for: for a statically highly intederminate system several equilibrated force systems can be found, and without flexibility data the "correct" one cannot be selected. There are several publications dealing with the indeterminacy of the results. Experiences show that though the order of the pairs greatly affect the individual contact forces and even the topology of the system, the overall, "macro" characteristics like average stress tensor or frequency diagram of contact force magnitudes remain the same, apart from slight statistical deviations.

8.5 Applications

The Contact Dynamics Method is rather popular among physicists studying granular dynamics problems (e.g. Daudon et al, 1997; Radjai et al, 1998; Unger et al, 2004), but not applied yet in the engineering practice, partly because of the doubts on its reliability and the random nature of the results.

Questions

8.1. Write the equations of motion of pair in CD, and explain the meaning of the quantities in it!
8.2. Describe the mechanics of the contacts in CD!
8.3. Explain the analysis of a time step in CD!