When Heyman's Safe Theorem of Rigid Block Systems Fails: Non-Heymanian Collapse Modes of Masonry Structures

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Abstract

Heyman's Safe Theorem is the theoretical basis for several calculation methods in masonry analysis. According to the theorem, the existence of an internal force system which equilibrates the external loads guarantees that the masonry structure is in a stable equilibrium state, assuming that a few conditions on the material behaviour are satisfied: the stone blocks have infinite compressional resistance, and the contacts between them resist only compression and friction. This paper presents simple examples in which the Safe Theorem fails: collapse occurs in spite of the existence of an equilibrated force system. A theoretical analysis of the stability of assemblies of rigid blocks with frictional contacts is then introduced: the virtual work theorem is derived, and a refined formulation of the Safe Theorem is given.

1. Introduction

Using the idea proposed by Kooharian (1952), a classic paper by Heyman (1966) suggested applying the concepts of Plastic Limit State analysis to masonry systems in cases when the collapse of the structure is caused by instability resulting from the unsuitable geometry of the structure. Heyman made the following assumptions about the material:

(i) Stone has no tensile strength.
(ii) The compressive strength of the stone is infinite.
(iii) Sliding of one stone on another cannot occur.

In addition, though not stated explicitly, Heyman also assumed that the blocks are rigid, so that the same given geometry of the structure is valid for any analysed force system which the structure is able to equilibrate.

Based on these assumptions, Heyman stated the Safe Theorem for masonry arches: "*If a line of thrust can be found which is in equilibrium with the external loads and which lies wholly within the masonry, then the structure is safe.*" (Kooharian (1952) stated the dual pair of the safe theorem, i.e. the Unsafe Theorem, in the following way: "*Collapse will occur (or will have occurred previously) if a kinematically admissible collapse state can be found. A "kinematically admissible" collapse state is one characterised by the condition that in a virtual displacement of the mechanism, the work done by the external loads must be at least as large as that done by the internal forces.*" The Unsafe Theorem, often referred to as the kinematic theorem, will not be considered in the present paper, which focuses purely on the Safe Theorem.

The Safe Theorem has been successfully applied in a vast number of engineering problems in practice, particularly to arches and vaults; in spite of the fact that neither Heyman nor

Kooharian gave a precise proof for these theorems. It was implicitly accepted that Plastic Limit Analysis can be directly applied to check whether the structure stands in the given geometry under its self weight, without making a distinction between the two types of problems which may occur in the stability analysis of arches and vaults: (1) the stability analysis of a structure which is already definitely stable under its selfweight and which is then loaded by an additional, one-parameter load whose magnitude starts to increase from zero, and whose allowable magnitude is to be determined, and (2) the stability analysis of a structure under a non-parametric load such as, for instance, its selfweight . (Speaking in the language of plastic limit analysis, for Problem (1) the zero load factor is within the domain of admissible load factors if the structure was found to be safe for its self weight.) While the application of Plastic Limit Analysis seems to be straightforward for Problem (1) if certain assumptions on the structural behaviour are met, the situation is different for Problem (2): if the structure is not in equilibrium for its selfweight, a stable state cannot be reached by proportionally decreasing the selfweight by any factor. The validity of the Safe Theorem for Problems (1) and (2) for masonry structures consisting of rigid blocks is the subject of the present study.

It will be useful to recognize how thinking on masonry has changed over the past few decades. Engineers in the 1950s and 1960s tended to think of masonry not as a collection of precisely described individual blocks, but rather as a kind of a continuum whose behaviour could be described with the help of homogenization methods, e.g. in soil mechanics. This view can be recognised in Heyman's approach and in that of several later authors such as Como (1992). The following decades, however, brought dramatic developments in both computational techniques and the hardware widely available to engineers, and this development led to masonry structures being considered rather as a collection of discrete, well-defined blocks, in which the displacement of each block should be analysed separately. A few examples of this discrete way of thinking are the Discrete Element Method (e.g. UDEC invented by P.A. Cundall, 1971; DDA by G.-H. Shi, 1988), funicular analysis proposed by O'Dwyer (1991), thrust network analysis in 2D (Block, 2005) and in 3D (Block and Ochsendorf, 2007). From the 1980s onwards it became numerically possible to simulate simple masonry structures by modelling the state of each individual block separately and by the 1990s real practical problems were frequently being solved with the help of discrete computational methods.

These developments also influenced views of Heyman's condition (i). In reality, stone or brick voussoirs do resist tension and joints (which are dry, or with very weak mortar) between blocks are the only parts of the system where it is reasonable to assume no-tension behaviour. While some authors (e.g. O'Dwyer, 1999, Huerta, 2001, J. A. Ochsendorf et al, 2004) continue to use the original formulation of the conditions, others (for instance Boothby, 2001 or D'Ayala and Tomasoni, 2008), thinking "discretely", replaced Heyman's assumption (i) by the requirement (perhaps physically more realistic on the scale of voussoirs) that only joints are no-tensional. Boothby (2001) formulated the basic assumptions of the Safe Theorem as follows:

(a) the masonry units are infinitely rigid;
(b) the masonry units are infinitely strong;
(c) the masonry units do not slide at the joints;
(d) the joints transmit no tension.

The Safe Theorem can now be stated thus: "*If there exists any system of forces satisfying (a-d) and being in equilibrium with the loads, then the structure is safe*." It was generally accepted without proof that the Safe Theorem, being successful in so many practical applications even for selfweight, would remain valid if Heyman's no-tension criterion for the material as a whole is replaced by a no-tension criterion for joints only (assuming, of course, that frictional sliding does not occur).

The occurrence of frictional sliding was excluded from the analyses of Kooharian, Heyman and the numerous authors who followed them, in order to ensure the validity of the normality condition, an indispensable hypothesis of classic limit state analysis. If normality fails the uniqueness theorem is no longer valid and nor are the static and kinematic theorems in their classic forms. Non-associated flow rules can be applied to plastic limit analysis in the absence of normality, although Orduna & Lourenco (2005a) emphasize that the static and kinematic approaches cannot be separated and that a mixed formulation must be used and thus a multiplicity of solutions can exist in these situations.

For masonry structures, the failure of normality in the case of Coulomb-type frictional contact sliding was demonstrated by Drucker as early as 1954. Parland (1982), (1995) also drew attention to this. In his contact model surface roughness forces the frictionally sliding contacts to dilate. Normality would be valid if the friction angle according to which the tangential force is related to the compression force at sliding, and the angle (expressing the surface roughness) according to which the normal deformation is related to the tangential deformation at sliding were equal. However, there is no physical reason to assume such a coincidence: the dilation angle is usually significantly lower.

The lack of normality of Coulomb-type contacts posed a challenging problem for several researchers aiming at developing reliable computational techniques for masonry analysis. In the limit state method of Livesley (1978, 1992) the problem was solved with the help of additional correcting steps. Orduna & Lourenco (2005a) applied a piecewise linear approximation of the yield surface for 3D analysis, including torsional failure of the planar contacts between voussoirs. They emphasize that the loading history of a structure can significantly influence the results of the analysis, and without knowledge of this history the reliability of the solutions is questionable. In their next paper (Orduna & Lourenco, 2005b) a solution method is presented which is based on an approximate simulation of the loading history. For axially symmetric structures and loads, Casapulla and D'Ayala (2001) gave a proof for the uniqueness of the solution and presented a computer procedure based on the static theorem. Later the method was extended (D'Ayala and Tomasoni, 2008) and by finding the optimal thrust surface vaults with more general shapes could be analysed.

These investigations and successful numerical techniques may give the reader the false impression that if frictional sliding is excluded from the possible behaviour of masonry structures then the plastic limit theorems are valid and that, more particularly, Heyman's Safe Theorem holds. In the present paper the simple examples given in Section 2 will demonstrate that even without the presence of frictional sliding the static theorem may give incorrect results. Section 3 introduces the theoretical background of the problem of the Safe Theorem, and argues that the theorem holds only for a limited range of displacement systems where the tangential component of the relative translations in the joints is zero everywhere. This analysis reveals that there are two theoretically different possible ways for a masonry structure to collapse in spite of the existence of an equilibrated force system for the analysed structure. Finally, Section 4 discusses the results, and presents two very simple examples which capture the essence of the two possibilities for collapse.

2. Examples when Heyman's Safe Theorem fails

Section 2 presents two-dimensional examples in which rigid block systems collapse in spite of the existence of an equilibrated force system, even though frictional sliding is not present at collapse. The failure of Heyman's Safe Theorem is evident in these examples, and even a very inexperienced engineer can easily recognise that the structures portrayed in Figs 1-3 will collapse. However, if a similar situation occurs somewhere hidden in a complex 3D structure, a computer analysis based on the classic Safe Theorem of masonry structures will find the analysed state to be stable, even though failure may occur. The aim of introducing these examples is to underline the necessity of finding an improved formulation of the Static Theorem for masonry structures.

Example 1: The Overloaded Roof
This example focuses on the traditional problem of plastic limit state analysis (Problem (1) in the Introduction). It shows that a structure which is initially in a stable equilibrium state may collapse as a result of increasing load, even though an equilibrated force system exists for the increased load.

The structure is shown in Figure 1a. The three vertical columns are fixed. Block 1 is loaded with its given selfweight G_1 and with the additional load G_2 exerted by the block in the upper right corner. Block 1 is in a stable equilibrium state for a zero or negligible G_2 for which the resultant $G \triangleq (G_1, G_2)$ acts along a vertical line on the left side of point P. As the load G_2 increases, the line of action of the resultant G gradually shifts to the right side of point P. Figure 1a shows that even in this case an equilibrated force system satisfying conditions (a-d) exists. However, collapse happens in the way shown in Figure 1b.

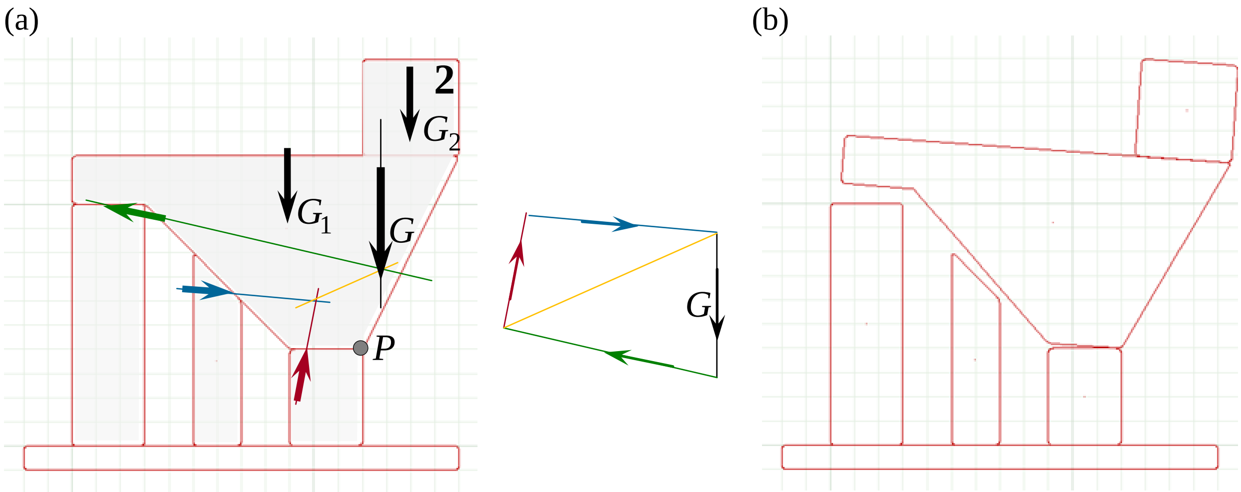

Figure 1.
The Overloaded Roof, (a) Force system equilibrating the loads, (b) collapse mechanism

Example 2: The Buckling Arch
This example corresponds to the classical arch problem of masonry analysis: the geometry of an arch is given (Figure 2a), and the question to decide is whether a structure with this geometry is able to balance its selfweight (referred to as Problem (2) in the Introduction).

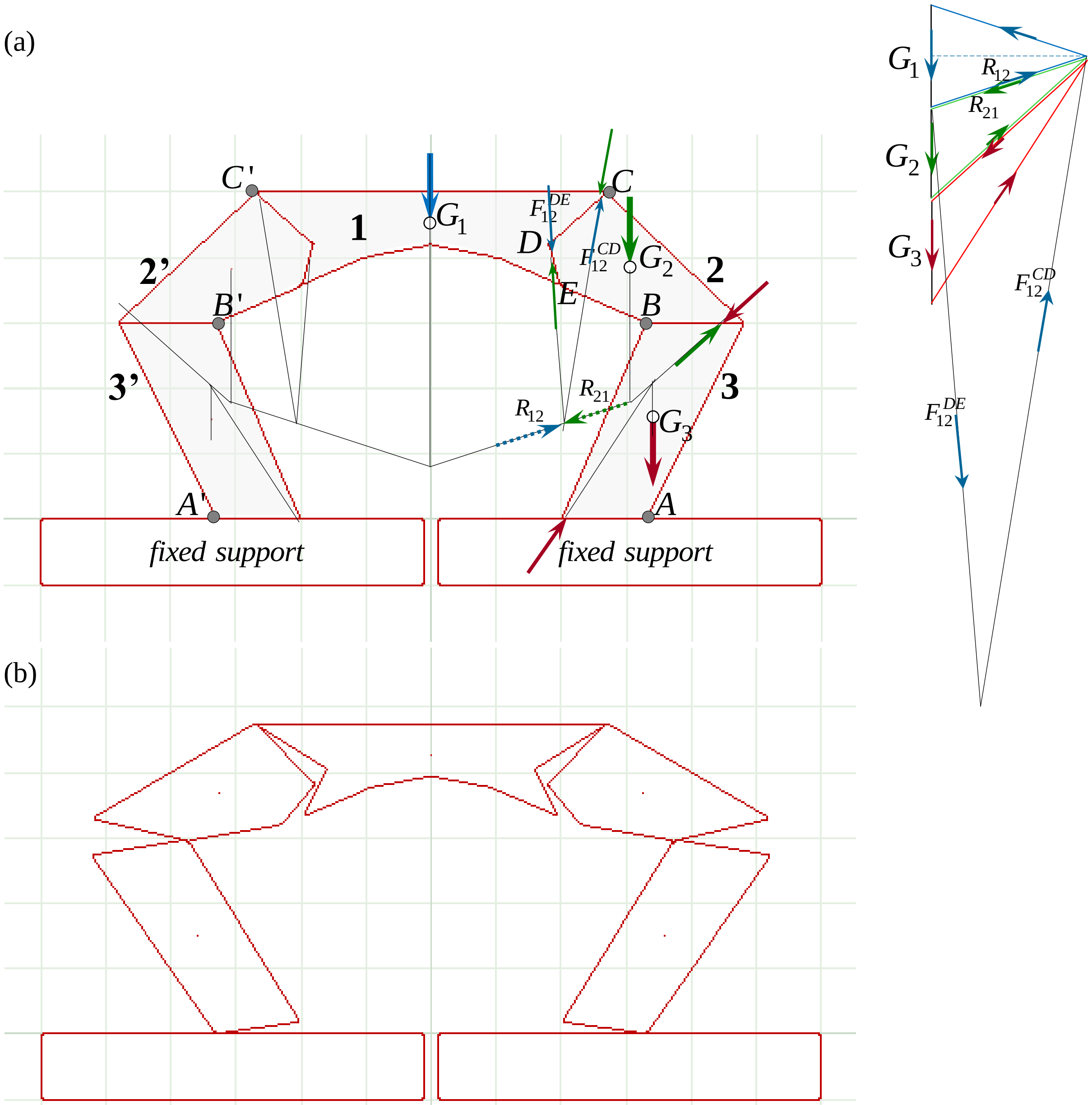

Figure 2.
The Buckling Arch, (a) Force system equilibrating the loads, (b) collapse mechanism

This structure consists of five blocks, each having equal selfweight arranged symmetrically about a vertical axis. A system of forces is shown in Figure 2a which satisfies conditions (a)-(d) and keeps the structure (all blocks, and all combinations of blocks) in equilibrium. (The force diagram is symmetric, hence for the sake of simplicity only the half of the force diagram is shown.)

An important feature of the structure is that the contact between Blocks 1 and 2 consists of two parts, *CD* and *DE*. In the force system presented here none of these partial contacts carry tension, but the resultant R_{12} is far outside Block 1 and consequently tension occurs in the material of the block.

Figure 2b shows the following displacement system:

- → The displacements are symmetric with respect to the vertical axis, so Block 1 does not rotate, and does not translate horizontally either. However, it will translate downwards, i.e. a purely vertical, identical translation happens in every point of Block 1.
- → Block 3 rotates in a clockwise direction about point *A*, and Block 3' moves in the opposite direction;
- → Block 2 is attached to Block 3 at their common point *B*, so this point of Block 2 translates outwards together with Block 3. In addition to this, Block 2 rotates about *B* by a counter-clockwise angle to ensure that point *C* does not translate horizontally. Block 2' moves in the opposite way.
- → Consequently, the vertical translations of the points where the selfweight forces act are downwards.

The structure collapses according to this displacement system, in spite of the existence of a statically admissible force system.

<u>Example 3.:</u> The Inclined Tower

This example illustrates that as a consequence of support displacements, a structure which is initially in a stable state is shifted into an unstable equilibrium, which is then erroneously found to be safe by the Static Theorem.

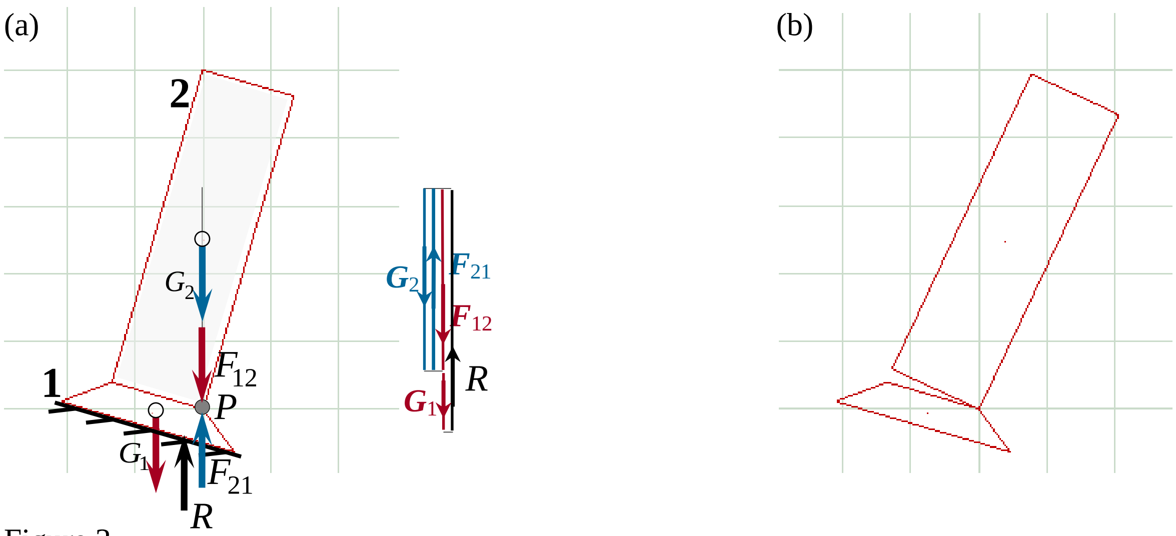

Figure 3.
The Inclined Tower, (a) Force system equilibrating the loads, (b) Collapse mechanism

The structure in Figure 3 consists of two blocks, the lower of which rests on an inclined foundation. For a horizontal or nearly-horizontal foundation the structure would be in a stable equilibrium state. As inclination increases, a geometry is reached as shown in Figure 3a: the line of action of G_2 now goes exactly through point *P* and consequently in any equilibrated force system the contact force between Blocks 1 and 2 must go through point *P*. Such an equilibrated force system is shown in Figure 3a. However, this is not a safe state: when subjected to a small perturbation the structure collapses as shown in Figure 3b.

Note that there is a basic difference between the examples given above: in Examples 1 and 2 the structure cannot be in equilibrium at all with the given geometry, while in Example 3 there is an equilibrium configuration, although this equilibrium is unstable.

Regarding Examples 1 and 2, the role of frictional resistance in the contacts has to be emphasized. In both cases the equilibrating force systems must contain contact forces which have a frictional component. Heyman's basic conditions assume infinite frictional resistance. Indeed, without these frictional components the loads could not be equilibrated and therefore the erroneous result that these structures are safe would not be produced by the theorem. This question will also be considered in the forthcoming theoretical analysis.

3. The Static Theorem

In Section 3 the static theorem for masonry structures consisting of rigid blocks with no-tension Coulomb-frictional contacts will theoretically be derived.

The forthcoming theoretical analysis was greatly inspired by the rigorous proof given by Como (1992), (2012) for no-tension continua. That proof is valid without considering any load factor or assuming a proportional change of the loads from small values where the structure is still in equilibrium up to large values where the structure already collapses. In that proof, however, the no-tension behaviour of the voussoirs is assumed: the definite requirement is posed there that tension cannot exist in any point of the blocks on any cut.

The analysis in the present paper does not include such a requirement: tension may exist in any parts of the blocks except for the contact surfaces which are assumed to be no-tension Coulomb-frictional contacts.

3.1 Equilibrium of a masonry system

<u>Geometry</u>
The structure analysed consists of perfectly rigid blocks (voussoirs) with planar contacts between them. A reference point is assigned to every block, which is perhaps (but not necessarily) the mass centre of the block. The vector $\mathbf{x}_b$ denotes the position of the reference point of the b-th block (see Figure 4). The translation of the reference point, together with the rotation of the block about this point, uniquely determines the displacements of any point of the block. The blocks are referred to by their indices b. The vector $\mathbf{n}_{bc}$ is the outwards unit normal vector of the surface of block b at its contact c.

In addition to the voussoirs that are able to translate and rotate, the structure also contains support elements: these are fixed blocks that cannot move. The indices of the support elements are set to be larger than the index of any movable block.

The contacts formed by any two blocks are referred to according to their contact index c. If two blocks, b_1 and b_2, form a contact, then the block with the smaller index is considered to be "the first block" of the contact and the other block (with the larger index) is "the second". Hence in a contact between a movable block and a support element the support is always "the second".

<u>Forces</u>

The blocks are perfectly rigid, and able to resist tension, so there is no restriction on the sign or magnitude of the principal stresses in any internal point of a block (the strains are zero). Conservative external static loads act on the blocks; they may be concentrated forces, or forces distributed within the volume of or along a part of the surface of the blocks. The external forces – but not those contact forces exerted by the neighbouring blocks, which will be dealt with a few rows below – are reduced to the mass centre, and produce a force and a moment ($\mathbf{G}_b$, $\mathbf{M}_b$), as shown in Figure 4.

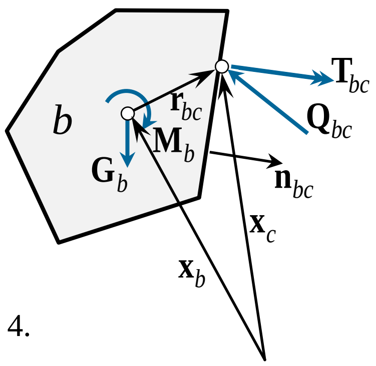

Figure 4.

A distributed force acts on block b along the contact surface, expressed by its neighbour on the other side of the contact. This distributed force consists of a normal and a tangential component both of which may vary from point to point along the contact. At any point, the normal component can only be compressional, while the direction of the tangential component is arbitrary in the plane of the contact. According to Coulomb's law of friction, at any point of the surface the magnitude of the tangential component does not exceed the magnitude of the normal component times the friction coefficient (f).

The resultant of the normal components acting on c is a compression force, $\mathbf{Q}_{bcN}$, whose point of action cannot be outside the contact area. The point of action of $\mathbf{Q}_{bcN}$ will have a particular importance: this will be referred to as "the contact point", whose position vector will be denoted by $\mathbf{x}_c$: this is either an internal or a boundary point of the contact surface. The vector pointing from the centre of b to the contact point is $\mathbf{r}_{bc}$. Note that the location of the contact point is not the geometrical characteristic of the structure: for different force systems acting on the same structure, different $\mathbf{x}_c$ and $\mathbf{r}_{bc}$ vectors will be found.

Let the tangential component of the distributed contact force be reduced to the contact point, resulting in a concentrated force vector $\mathbf{Q}_{bcT}$ parallel to the contact surface, and a moment vector $\mathbf{T}_{bc}$ which is perpendicular to the contact surface. Since the distributed tangential forces obey the Coulomb relation, $|\mathbf{Q}_{bcT}| \le f \cdot |\mathbf{Q}_{bcN}|$, but in the presence of a nonzero torsional moment $|\mathbf{Q}_{bcT}|$ cannot reach the value $f \cdot |\mathbf{Q}_{bcN}|$.

Using these notations, the equilibrium of block b is expressed by the following equations:

$$\mathbf{G}_b + \sum_{(bc)} \mathbf{Q}_{bc} = 0 \tag{1}$$

$$\mathbf{M}_b + \sum_{(bc)} \left(\mathbf{r}_{bc} \times \mathbf{Q}_{bc} + \mathbf{T}_{bc} \right) = 0 \tag{2}$$

The summation for index bc goes along the contacts of block b. The symbol $\times$ stands for the vectorial product.

The total structure consists of several blocks each of which has to be in equilibrium. Some of the blocks are in contact with non-displacing supports; the forces expressed by these contacts on the movable blocks are treated according to exactly the same manner.

The external forces and moments can be collected into a hypervector ($\mathbf{G}_1$, $\mathbf{M}_1$; $\mathbf{G}_2$, $\mathbf{M}_2$; …), symbolically denoted by ($\mathbf{G}$, $\mathbf{M}$). Similarly, the contact forces and torsions for the hypervector ($\mathbf{Q}_1$, $\mathbf{T}_1$; $\mathbf{Q}_2$, $\mathbf{T}_2$; …) are shortly written ($\mathbf{Q}$, $\mathbf{T}$).

Virtual displacement systems

The degrees of freedom of the structure are the translation and rotation of all the blocks; a system of virtual displacements is uniquely defined by specifying the infinitesimally small $\delta\mathbf{u}_b$ translations of the centres of blocks and the $\delta\boldsymbol{\varphi}_b$ rotations of the blocks about their centres. These vectors are collected into the hypervector ($\delta\mathbf{u}$, $\delta\boldsymbol{\varphi}$), which consists of as many $\delta\mathbf{u}_b$ and $\delta\boldsymbol{\varphi}_b$ vectors as the number of voussoirs in the structure.

Since the displacements are small, the translation of a contact point bc on block b can be calculated as:

$$\delta\mathbf{u}_{bc} = \delta\mathbf{u}_b + \delta\boldsymbol{\varphi}_b \times \mathbf{r}_{bc} \tag{3}$$

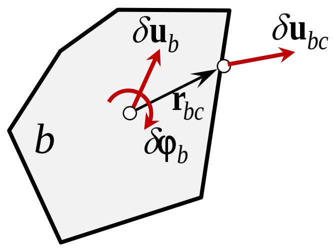

Figure 5.

The blocks b_1 and b_2 form contact c so that $b_1 < b_2$: as mentioned above, by definition the block with the smaller index is considered as “the first block” of the contact and the other is “the second”. The virtual relative translation $\delta\mathbf{d}_c$ assigned to the contact point is defined as the relative translation of the contact point on the first block with respect to the second block. Similarly, the virtual relative rotation $\delta\boldsymbol{\theta}_c$ of the contact is understood as the rotation of the first block relative to the second block. The translational and rotational deformation of the contact is:

$$\begin{aligned}
\delta\mathbf{d}_c &= \delta\mathbf{u}_{b_1c} - \delta\mathbf{u}_{b_2c} = \delta\mathbf{u}_{b_1} + \delta\boldsymbol{\varphi}_{b_1} \times \mathbf{r}_{b_1c} - \delta\mathbf{u}_{b_2} + \delta\boldsymbol{\varphi}_{b_2} \times \mathbf{r}_{b_2c} \\
\delta\boldsymbol{\theta}_c &= \delta\boldsymbol{\varphi}_{b_1} - \delta\boldsymbol{\varphi}_{b_2}
\end{aligned} \tag{4}$$

For those contacts which are formed between a block and a support, the same definition holds, but since for the support (i.e. for the second entity of the contact) the translation of any point is zero, the virtual relative translation belonging to the contact has a simpler form:

$$\begin{aligned}
\delta\mathbf{d}_c &= \delta\mathbf{u}_{b_1c} - \delta\mathbf{u}_{b_2c} = \delta\mathbf{u}_{b_1} + \delta\boldsymbol{\varphi}_{b_1} \times \mathbf{r}_{b_1c} \\
\delta\boldsymbol{\theta}_c &= \delta\boldsymbol{\varphi}_{b_1}
\end{aligned} \tag{5}$$

These contact deformations are collected into the ($\delta\mathbf{d}$, $\delta\boldsymbol{\theta}$) virtual deformation vector, a hypervector containing as many $\delta\mathbf{d}_c$ and $\delta\boldsymbol{\theta}_c$ vectors as the number of contacts in the analysed structure.

Note that apart from being infinitesimally small, no other restrictions apply for the chosen translations and rotations; correspondence to the correct mechanical behaviour of the masonry system is not required at all. Hence, in the system of virtual displacements the blocks can e.g. penetrate into each other, translate along each other in any tangential direction independently of the magnitude and direction of frictional forces acting at the contacts etc.

The Theorem of Virtual Displacements

Now the equilibrium equations (1) and (2) will be transformed into another form. Eqs. (1) are true if and only if for any arbitrarily chosen virtual translations of the blocks, the scalar equations

$$\mathbf{G}_b \cdot \delta\mathbf{u}_b + \sum_{(bc)} \left(\mathbf{Q}_{bc} \cdot \delta\mathbf{u}_b \right) = 0 \tag{6a}$$

hold for every block. (Here the symbol · denotes scalar product.) Similarly, eqs. (2) are true if and only if for any virtual rotations of the blocks

$$\mathbf{M}_b \cdot \delta\boldsymbol{\varphi}_b + \sum_{(bc)} \left(\left(\mathbf{T}_{bc} + \mathbf{r}_{bc} \times \mathbf{Q}_{bc} \right) \cdot \delta\boldsymbol{\varphi}_b \right) = 0 \tag{6b}$$

holds. Considering the whole system of blocks and summing up the above equations according to b, the sufficient and necessary condition of the equilibrium is that for any arbitrarily chosen system of virtual displacements ($\delta\mathbf{u}$, $\delta\boldsymbol{\varphi}$):

$$\sum_{(b)} \left(\mathbf{G}_b \cdot \delta\mathbf{u}_b \right) + \sum_{(b)} \left(\sum_{(bc)} \left(\mathbf{Q}_{bc} \cdot \delta\mathbf{u}_b \right) \right) + \sum_{(b)} \left(\mathbf{M}_b \cdot \delta\boldsymbol{\varphi}_b \right) + \sum_{(b)} \left(\sum_{(bc)} \left(\left(\mathbf{T}_{bc} + \mathbf{r}_{bc} \times \mathbf{Q}_{bc} \right) \cdot \delta\boldsymbol{\varphi}_b \right) \right) = 0. \tag{7}$$

Using the identity $\left(\mathbf{a} \times \mathbf{b} \right) \cdot \mathbf{c} = \mathbf{b} \cdot \left(\mathbf{c} \times \mathbf{a} \right)$, the last term on the left side of this scalar equation can be rearranged:

$$\sum_{(b)} \left(\mathbf{G}_b \cdot \delta\mathbf{u}_b \right) + \sum_{(b)} \left(\mathbf{M}_b \cdot \delta\boldsymbol{\varphi}_b \right) + \sum_{(b)} \left(\sum_{(bc)} \left(\mathbf{Q}_{bc} \cdot \delta\mathbf{u}_b \right) \right) + \sum_{(b)} \left(\sum_{(bc)} \left(\mathbf{T}_{bc} \cdot \delta\boldsymbol{\varphi}_b + \mathbf{Q}_{bc} \cdot \left(\delta\boldsymbol{\varphi}_b \times \mathbf{r}_{bc} \right) \right) \right) = 0 \tag{8a}$$

or equivalently:

$$\sum_{(b)} \left(\mathbf{G}_b \cdot \delta\mathbf{u}_b + \mathbf{M}_b \cdot \delta\boldsymbol{\varphi}_b \right) + \sum_{(b)} \left(\sum_{(bc)} \left(\mathbf{Q}_{bc} \cdot \left(\delta\mathbf{u}_b + \delta\boldsymbol{\varphi}_b \times \mathbf{r}_{bc} \right) \right) \right) + \sum_{(b)} \left(\sum_{(bc)} \left(\mathbf{T}_{bc} \cdot \delta\boldsymbol{\varphi}_b \right) \right) = 0 \tag{8b}$$

On the left side the second term is a summation along the contacts in the system. Those contacts which are formed by two blocks (denote them by i from now, referring to "internal" contacts) are considered twice (i.e. once for the first block and once for the second block of the contact), while the contacts between a block and a support (denoted below by e, for "external" contacts) are taken into account once (the block is indeed considered, but the support whose virtual displacements are zero does not take part). Eq. (8) can be rearranged:

$$\begin{aligned}&\sum_{(b)}\left(\mathbf{G}_b\cdot\delta\mathbf{u}_b+\mathbf{M}_b\cdot\delta\boldsymbol{\varphi}_b\right)+\sum_{(i)}\left(\mathbf{Q}_{b_1i}\cdot\left(\delta\mathbf{u}_{b_1}+\delta\boldsymbol{\varphi}_{b_1}\times\mathbf{r}_{b_1i}\right)+\mathbf{Q}_{b_2i}\cdot\left(\delta\mathbf{u}_{b_2}+\delta\boldsymbol{\varphi}_{b_2}\times\mathbf{r}_{b_2i}\right)\right)+\\&+\sum_{(i)}\left(\mathbf{T}_{b_1i}\cdot\delta\boldsymbol{\varphi}_{b_1}+\mathbf{T}_{b_2i}\cdot\delta\boldsymbol{\varphi}_{b_2}\right)+\sum_{(e)}\left(\mathbf{Q}_{b_1e}\cdot\left(\delta\mathbf{u}_{b_1}+\delta\boldsymbol{\varphi}_{b_1}\times\mathbf{r}_{b_1e}\right)\right)+\sum_{(e)}\left(\mathbf{T}_{b_1e}\cdot\delta\boldsymbol{\varphi}_{b_1}\right)=0\end{aligned}\tag{9}$$

For simplicity, introduce a new notation for the contact forces: instead of the two opposite forces $\mathbf{Q}_{b1i}$ and $\mathbf{Q}_{b2i}$ belonging to the same contact i, the contact force $\mathbf{Q}_i$ is, by definition, the force acting on the block with the smaller index (the same can be done for the torque):

$$\begin{aligned}\mathbf{Q}_i&:=\mathbf{Q}_{b_1i}\ ;\quad \mathbf{Q}_{b_2i}=-\mathbf{Q}_i\\ \mathbf{T}_i&:=\mathbf{T}_{b_1i}\ ;\quad \mathbf{T}_{b_2i}=-\mathbf{T}_i\end{aligned}\tag{10}$$

For external contacts, the force and torque acting on the first entity of the contact is considered only, so $\mathbf{Q}_e := \mathbf{Q}_{b1e}$ and $\mathbf{T}_e := \mathbf{T}_{b1e}$. Using these notations, Eq. (9) can shortly be written as follows:

$$\begin{aligned}&\sum_{(b)}\left(\mathbf{G}_b\cdot\delta\mathbf{u}_b+\mathbf{M}_b\cdot\delta\boldsymbol{\varphi}_b\right)+\sum_{(i)}\mathbf{Q}_i\left(\left(\delta\mathbf{u}_{b_1}+\delta\boldsymbol{\varphi}_{b_1}\times\mathbf{r}_{b_1i}\right)-\left(\delta\mathbf{u}_{b_2}+\delta\boldsymbol{\varphi}_{b_2}\times\mathbf{r}_{b_2i}\right)\right)+\\&\sum_{(c)}\mathbf{T}_c\left(\delta\boldsymbol{\varphi}_{b_1}-\delta\boldsymbol{\varphi}_{b_2}\right)+\sum_{(e)}\left(\mathbf{Q}_e\cdot\left(\delta\mathbf{u}_{b_1}+\delta\boldsymbol{\varphi}_{b_1}\times\mathbf{r}_{b_1e}\right)\right)+\sum_{(e)}\left(\mathbf{T}_e\cdot\delta\boldsymbol{\varphi}_{b_1}\right)=0\end{aligned}\tag{11}$$

The virtual deformations (4) and (5) belonging to the contacts can now be recognized in (11), and hence

$$\sum_{(b)}\left(\mathbf{G}_b\cdot\delta\mathbf{u}_b+\mathbf{M}_b\cdot\delta\boldsymbol{\varphi}_b\right)+\sum_{(c)}\left(\mathbf{Q}_c\cdot\delta\mathbf{d}_c+\mathbf{T}_c\cdot\delta\boldsymbol{\theta}_c\right)=0\ .\tag{12}$$

The first sum whose index b runs along the blocks expresses the external virtual work, done by the given loads on the chosen virtual displacements of the blocks. The second sum (with index c running along all internal and external contacts) is the internal virtual work, done by the contact forces and torques on those contact deformations that are caused by the chosen virtual block displacements.

Equation (12) formulates the principle of virtual displacements for assemblies of rigid blocks:

Theorem #1.:

> A force system composed by the (**G**, **M**) reduced loads acting on the $\mathbf{x}_b$ block centres and the (**Q**, **T**) contact forces acting at the $\mathbf{x}_c$ points is an ***equilibrium system*** if and only if for any arbitrary ($\delta\mathbf{u}$, $\delta\boldsymbol{\varphi}$) virtual displacements of the blocks and corresponding ($\delta\mathbf{d}$, $\delta\boldsymbol{\theta}$) virtual contact deformations the sum of the external and internal virtual work is zero.

3.2 Mechanically admissible displacement systems: Heymanian and non-Heymanian displacements

3.2.1 Mechanically admissible virtual displacements

Rigid blocks cannot interpenetrate into each other, so the normal component of the relative translations at any point along a contact surface may only mean separation but no overlap, if the displacement system obeys the rigidity assumption of the blocks. Virtual displacement systems do not necessarily satisfy this no-penetration criterion, hence from now the special name "mechanically admissible" will be given to those virtual displacement systems which obey the no-penetration requirement. Denote the normal component of $\delta\mathbf{d}_c$ by $\delta\mathbf{d}_{cN}$. Since the contacts do not resist tension, the following relation holds for every contact in the case of such displacement systems:

$$\mathbf{Q}_{cN}\cdot\boldsymbol{\delta}\mathbf{d}_{cN}\geq 0\ . \qquad (13)$$

Virtual displacement systems ($\delta\mathbf{u}$, $\delta\boldsymbol{\varphi}$) for which (13) is valid for all systems of no-tension contact forces and for all contacts, will be called, by definition, *mechanically admissible virtual displacement systems*. (Note that since (13) must hold for any system of no-tension contact forces, (13) means that in *every* point of the contact surfaces only separation or zero relative normal displacement – and no interpenetration – may occur.)

The set of mechanically admissible virtual displacement systems can be separated into two subsets, i.e. Heymanian and non-Heymanian systems, according to whether a tangential component of the relative translation exists in any point of any contact. A displacement system is said to be *Heymanian* if the tangential components of the relative translations ($\delta\mathbf{d}_c$) and the normal components of the relative rotation vectors ($\delta\boldsymbol{\theta}_c$) are zero at every point of every contact surface. Figure 6 shows three possibilities: (a) pure relative translation in the normal direction; (b) relative rotation about a corner; and (c) the combination of the two: separation in normal direction together with relative rotation. On the other hand, if a tangential component of $\delta\mathbf{d}_c$ or a normal component of $\delta\boldsymbol{\theta}_c$ exists anywhere among the contacts in the structure, the displacement system is called *non-Heymanian*. As illustrated in Figure 7, such a contact deformation is not necessarily accompanied by frictional sliding: it may also occur while the two blocks are completely separate from each other. Figure 7. shows several different possibilities: relative translation without or together with contact separation (7a and 7b); relative rotation around an axis in the contact plane, accompanied by contact sliding (7c) and also by contact separation (7d); relative torsional rotation without (7e) or together with (7f) contact separation and sliding.

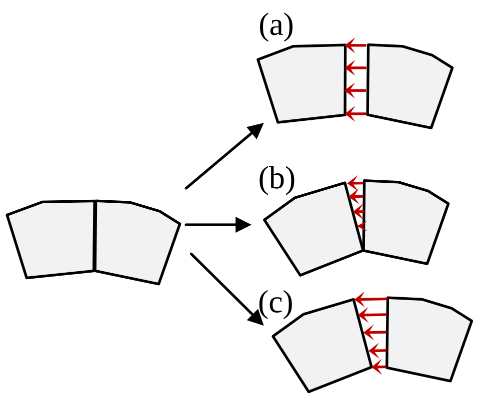

Figure 6.
Heymanian contact displacements

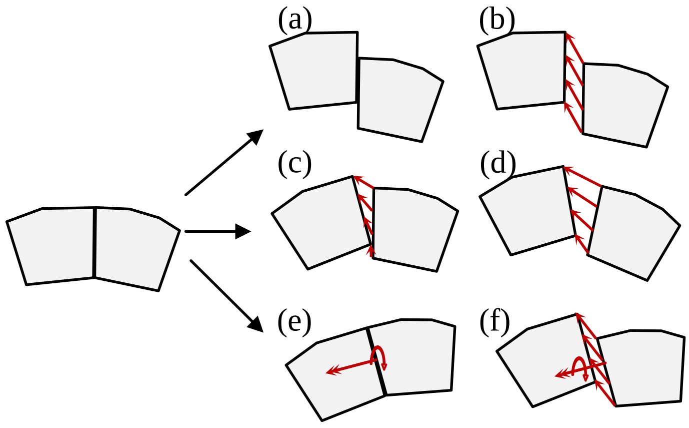

Figure 7.
Non-Heymanian contact displacements

3.2.2 Work done on a mechanically admissible virtual displacement system

Consider now the virtual external and internal work on the left side of Equation (12), and focus on the sign of the second term. If our analysis is restricted to Heymanian displacement systems only, then (13) ensures that for any arbitrary contact force system (obeying the no-tension criterion) this second term cannot be negative. It is definitely positive if there exists at least one contact in which the $\mathbf{x}_c$ contact point opens up in the chosen virtual displacement system; and it is zero if none of the contacts open up in the contact points. Consequently, the first term cannot be positive, independently of the exact details of the contact force system; and two possibilities may occur:

(i) If an equilibrium force system can be found for which for any nonzero Heymanian virtual displacement system at least one contact point opens up, then the existence of the equilibrated force system proves that the external work is negative on any arbitrarily chosen mechanically admissible Heymanian system. (It is important to emphasize again that the locations of the "contact points" are not the geometric characteristic of the structure: these are the points where the chosen contact forces act.)

(ii) However, if there exists any mechanically admissible Heymanian virtual displacement system for which the structure moves with none of the contact points opening up, then the work of the external forces done on this displacement system is zero.

An example for case (ii) was shown in Figure 3. Obviously, in any equilibrated force system the contact force between Blocks 1 and 2 must go through point P. Consider a mechanically admissible virtual displacement system which consists of zero displacements for Block 1, and an infinitesimally small rotation of Block 2 about point P. The internal virtual work is zero now, so in accordance to this, the external work should also be zero because of (12). Indeed, the small rotation of Block 2 causes a horizontal translation of its reference point, on which the work of the G_2 vertical force is zero. (Note that there are several mechanically admissible displacement systems for which the lower or the upper contact opens up, and for these systems the external work is negative because of (12) and (13), but there also exists a system for which the external work is equal to zero.)

It is important to emphasize that for non-Heymanian systems the internal virtual work may be either positive or negative, so no conclusion can be drawn about the sign of the external virtual work.

The role of frictional resistance in the contacts should also be emphasized here. Note that without the existence of tangential force components and twisting moments the internal work cannot be negative for any mechanically admissible virtual displacement system. Consequently the external work cannot be positive, irrespective of whether the displacement system is Heymanian or non-Heymanian, if an equilibrium system of forces exists without friction. The existence of frictional components is what makes the sign of the total work ambiguous for non-Heymanian displacement systems.

3.2.3 Mechanically admissible finite displacements

In the forthcoming stability analysis in Section 3.3 finite displacements will be considered: the stability of an analysed state will be decided according to the sign of the work done by the external and internal forces along finite displacements which perturbate the actual position of the structure. The displacements will be finite in the sense that instead of the first-order approximations in (4) and (5), the contact deformations and the displacements of the characteristic points should be derived with the help of exact geometrical relations (i.e. points of rotating bodies move along circular paths), but they will be assumed to be sufficiently small not to cause the creation of new contacts between initially non-contacting elements. Crack opening or contact separation is, however, possible, and the contact force and torque which act at an existing contact in the analysed state, will either change position, or disappear in these cases. The two possibilities are illustrated in Figure 8.

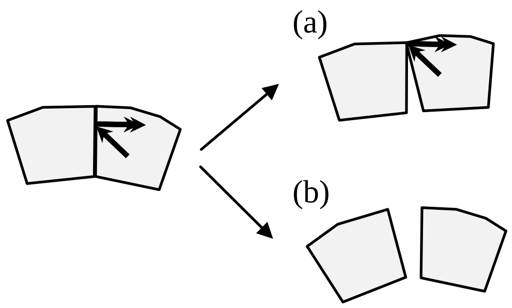

Figure 8.
Contact opening possibilities: Case (a): Crack opening with the blocks remaining in contact, Case (b): Contact separation with complete detachment of the blocks. Thick black arrows represent the resultant contact force and torsion moment exerted by the right block on the left block.

The ($\Delta\mathbf{u}$, $\Delta\boldsymbol{\varphi}$) finite displace system is *mechanically admissible* if
(a) the boundary points of the blocks do not penetrate into the interior of any other blocks, and
(b) tangential or torsional sliding happens only if the corresponding friction force or torsion moment actually acting in the contact has reached its Coulomb-limit in the direction opposite to the contact deformation.
Note that for mechanically admissible finite displacements the internal work in a sliding contact is always negative.

Finally, a system of finite displacements is called *Heymanian* or *non-Heymanian* according to whether its first-order approximation is Heymanian or non-Heymanian, respectively.

3.2.4 The sign of the work done along mechanically admissible finite displacements

External work

It was shown in Section 3.2.2 that for those structures for which nonzero Heymanian virtual displacement systems exist only when at least one contact opens up (either in the way shown in Fig. 8a, or as illustrated in Fig. 8b), the existence of an equilibrated force system proves that the external work is negative on any arbitrarily chosen mechanically admissible Heymanian virtual system. Since the first order approximation of any mechanically admissible Heymanian finite displacement system is a virtual displacement system of this kind, for sufficiently small (but still finite) displacements the external work is negative. This conclusion will be very important for the forthcoming stability analysis.

It was also pointed out that in cases when at least one mechanically admissible Heymanian virtual displacement system exists by which the structure moves without any contact opening up, the work of the external forces upon this infinitesimally small displacement system is zero. For such structures no conclusion can be drawn about the sign of the external work along finite displacements. Such a structure was shown in Figure 3. Obviously, for any sufficiently small (but finite) mechanically admissible rotation of Block 1 about its left or right bottom corner G_1 produces negative work. But for Block 2 the situation is different: consider a finite displacement system which contains zero displacements for Block 1, and a small clockwise rotation of Block 2 about its lowest point, *A*. The reference point of Block 2 starts to move along the horizontal tangent line of a circular path about the contact point. First-order approximation would give zero external work, but higher-order analysis reveals that the external work done on the chosen finite displacement system is positive.

Internal work

For mechanically admissible Heymanian systems of finite displacements the internal work (i.e. the work done by the existing internal forces and torques along the contact deformations) is always zero. A contact may (i) remain unchanged, (ii) open up in such a way that the boundary of the contact still remains in touch, or (iii) it may completely be opened. In the first case the contact force and torque work on zero deformations. In the second case (Fig. 8a) the contact force gets shifted (in a direction parallel to the contact plane) into a new position where the contact remains closed, so the internal work is zero again. In the third case (Fig. 8b) the contact force and torque disappears as soon as the contact starts to open up, so their work is zero.

For mechanically admissible non-Heymanian systems of finite displacements the internal work of the contact forces and torques may be either negative or zero: negative for those contacts where sliding (translational or torsional) occurs, and zero for contacts which remain unchanged, are cracked, or which completely open up. Without a frictional force component and twisting moment, however, the internal work is exactly zero.

3.3 Stability analysis

3.3.1 Definitions

Stability is a concept that has been inspiring intense scientific debates for more than a century. There exists no general agreement on what to mean by 'the' stability of a process or of a state; Szebehely (1984), for instance, collected nearly 50 different concepts of stability that are applied in dynamics and celestial mechanics. Regarding the stability of a solid or a structure, the situation is similar: as emphasized for instance by Belytschko et al. (2000), there exist several different definitions for stability: "stability is a concept that depends on the observer and his objectives". Consequently, every author has a significant freedom to settle his/her own definition for stability, but the concept must clearly and exactly be defined.

In accordance to the usual engineering approach and particularly corresponding to the rigorous criteria of the theory presented by Parland (1995), in the present paper the following definitions are used:

(1) Stable equilibrium state:
The actual state of a masonry system is *stable* if there exists a continuous, finite-sized domain of mechanically admissible finite displacement systems ($\Delta\mathbf{u}$, $\Delta\boldsymbol{\varphi}$) containing ($\Delta\mathbf{u} = \mathbf{0}$, $\Delta\boldsymbol{\varphi} = \mathbf{0}$) as an interior point, for which the total work done by the actual external and internal forces on any ($\Delta\mathbf{u}$, $\Delta\boldsymbol{\varphi}$) of the set is negative.

(2) Unstable equilibrium state:
The actual state of a masonry system is *unstable* if there exists any mechanically admissible finite displacement system ($\Delta\mathbf{u}$, $\Delta\boldsymbol{\varphi}$) for which the total work done by the actual external and internal forces on ($\alpha\cdot\Delta\mathbf{u}$, $\alpha\cdot\Delta\boldsymbol{\varphi}$) is positive for any $\alpha < 1$ multiplier.

(3) Neutral equilibrium state:
The actual state of a masonry system is *neutral* if it is not unstable, and there exists any mechanically admissible finite displacement system ($\Delta\mathbf{u}$, $\Delta\boldsymbol{\varphi}$) for which that the total work done by the actual forces on ($\alpha\cdot\Delta\mathbf{u}$, $\alpha\cdot\Delta\boldsymbol{\varphi}$) is zero for any $\alpha < 1$ multiplier.

(4) Critical equilibrium state:
The actual state of a masonry system is *critical* if there exists any mechanically admissible virtual displacement system ($\delta\mathbf{u}$, $\delta\boldsymbol{\varphi}$) for which the total virtual work of the actual external and internal forces is zero. (In this case higher-order analysis can reveal whether the actual state is stable, unstable or neutral.)

3.3.2 The Static Theorem

Assume now that for a given structure with given external loads, a force system was found which gives equilibrium with the loads, and obeys the no-tension criterion in the contacts. This "trial" force system may be very different from the forces actually acting in the structure, i.e. from the "real" force system. Though the trial force system is precisely known, the real forces are usually unknown (and not unique in a statically indeterminate structure like most masonry constructions). However, two characteristics of the real system can be recognised: first, the normal components of the contact forces are zero or compressional; and second, contact sliding (translational or rotational) is possible only if the distributed tangential contact forces along the contact surface are opposite to the relative tangential translation in every point of the surface.

The data of the trial force system contain the locations of the "contact points", i.e. the points where the $\mathbf{Q}_{cN}$ compression forces act. Depending on whether there is any possibility to introduce a mechanically admissible Heymanian virtual displacement system in such a way that none of the contact points open up, the following two cases may occur:
$\rightarrow$ If such a possibility does not exist (so at least one contact point always opens up, in every case), then the existence of the equilibrated trial force system ensures that for any arbitrary mechanically admissible Heymanian virtual displacement system the external work is negative. Since the first-order approximation of any finite displacements is a virtual displacement system, the external work is always negative for sufficiently small, mechanically admissible finite Heymanian systems. As discussed in Section 3.2.4, the internal work is zero for these finite displacements, so the total work is negative. Consequently, the structure is in a stable equilibrium state provided that non-Heymanian displacements are excluded from the analysis. On the other hand, no protection is given against non-Heymanian collapse modes.
$\rightarrow$ If there exists any possibility to perform nonzero mechanically admissible Heymanian virtual displacements without contact opening (i.e. if the structure can slightly be moved while all the $\mathbf{x}_c$ contact points remain closed), then along such an (infinitesimal) displacement system the external work is also zero, even though there may exist several other displacement systems for which the external work is negative. Consequently, no conclusions can be drawn concerning the sign of the external work along finite displacements, hence higher-order analysis is necessary to check whether Heymanian collapse modes exist. Even for Heymanian collapse modes, the state of the structure may be stable, unstable or neutral.

The trial force system does not give any hint on whether tangential relative displacements happen in the contacts. Regarding the real forces, for a mechanically admissible finite non-Heymanian system the internal work is always negative if sliding happens anywhere; but the sign of the sum of external and internal work cannot be predicted from the existence of a trial force system. So the structure may collapse according to a non-Heymanian collapse mode even if an equilibrium force system was found.

To summarize, for structures satisfying the following assumptions:
(a) the masonry units are infinitely rigid;
(b) the masonry units are infinitely strong;
(c) the joints transmit no tension, but resist arbitrary friction,
the Safe Theorem can be stated as follows:

Theorem #2.:

> If there exists any system of forces satisfying (a-c) being in equilibrium with the loads, and if there does not exist any mechanically admissible Heymanian virtual displacement system for which all contact points of this force system remain closed, then the structure is safe against collapse along any Heymanian displacements.

4. Discussion

Section 2 contained three examples (Figs. 1, 2, and 3) in which Heyman's classic static theorem failed. In the light of the results of Section 3, the reasons can now be explained:

→ The structures in Figures 1 and 2 are safe against Heymanian collapse modes: the existence of an equilibrated force system guarantees that without tangential relative displacements in the contacts, the structures will not collapse. Indeed, collapse happens in these cases with tangential relative displacements in certain contacts: non-Heymanian collapse occurs. (Note that the equilibrating force systems shown in Figures 1 and 2 contain frictional force components.)

→ The structure in Figure 3 is in a geometrical position for which, in order for it to have an equilibrated force system, the contact force between Blocks 1 and 2 must go through point P, so it is not possible to find any other $\mathbf{x}_c$ contact point between the two blocks. A mechanically admissible Heymanian virtual displacement system can be found, namely the clockwise rotation of Block 2 about P, for which no contact point opens up. Hence, the existence of an equilibrium force system does not guarantee that the sign of the external work is negative. Indeed, higher-order considerations immediately reveal that the structure cannot be in a stable equilibrium state: the external work is positive for a finite clockwise rotation of Block 2.

The two possible failure modes which may occur for frictional structures even if an equilibrated force system exists may be illustrated by two elementary examples, shown in Figures 9a and 9b:

<u>Boy with the Backpack</u>

The essence of the problem in Examples 1 and 2 is captured by the witty example proposed by P. Várkonyi (2012), which originated from Or (2007). As shown in Figure 9a, the "structure" consists of a rigid body in the shape of a sitting boy with a heavy backpack, supported by a brick-shaped rigid bench. The force G is the weight of the boy together with his backpack; its line of action is definitely outside the domain where the boy is supported from below. Figure 4a shows an equilibrium system in which both contact forces, F_1 and F_2, are compressional, having a frictional component. Although an equilibrium system exists, the body does not remain in the given position: it will fall over, rotating about point A. Collapse happens according to a non-Heymanian displacement system.

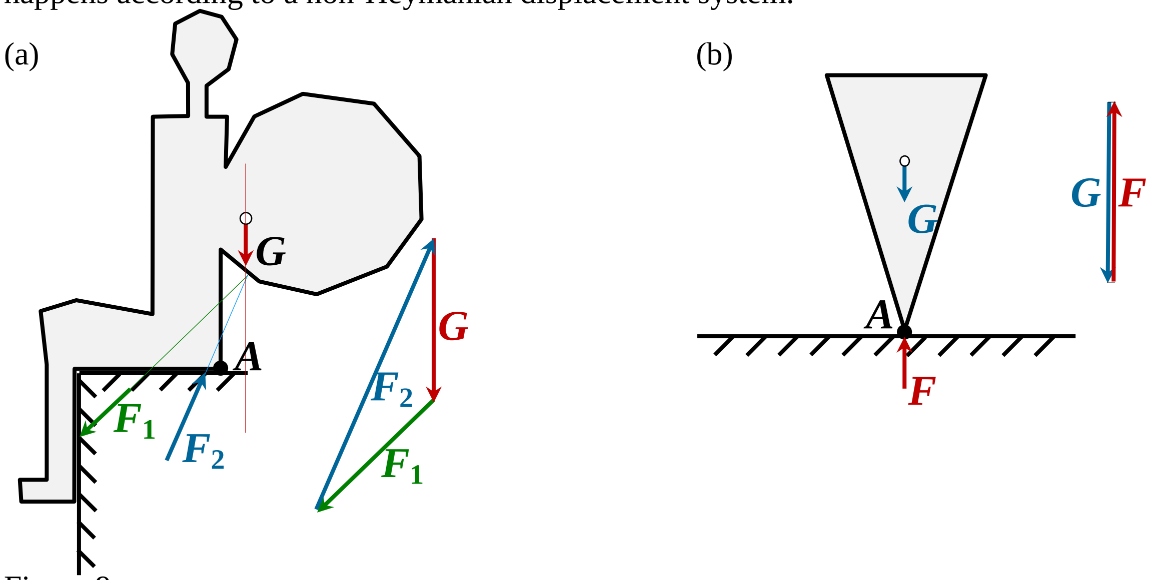

Figure 9.
(a) Boy with the Backpack; (b): Inverted Pyramid

Inverted Pyramid

The structure shown in Figure 9b consists of a triangular block which is supported at a single point (*A*). Obviously, with suitable vertical contact force acting at *A*, an equilibrated force system is easy to find. However, as the total work done by the external and internal forces on a small finite rotation about *A* is positive, the triangular block will topple over about point *A*.

Note that there is a basic difference between the two examples: in Fig. 9a the structure cannot be in equilibrium at all with the given geometry, while in Fig. 9b an equilibrium configuration exists though this equilibrium is unstable.

Finally it is important to note the role of the frictional component of the contact forces and moments in the trial equilibrium system. These components play a fundamental role in the failure of Heyman's Safe Theorem. It was pointed out at the end of Section 3.2.2 that without the existence of these frictional components the external work done on any mechanically admissible virtual (infinitesimally small) displacement system cannot be positive and consequently the external work is zero or negative also for any sufficiently small but finite mechanically admissible displacement system, regardless of whether they are Heymanian or non-Heymanian. This suggests an alternative formulation of the Safe Theorem:

Theorem #3.:

If there exists any system of forces satisfying the following conditions:
(i) the masonry units are infinitely rigid;
(ii) the masonry units are infinitely strong;
(iii) the joints transmit no tension and no friction
being in equilibrium with the loads, and if there does not exist any mechanically admissible virtual displacement system for which all contact points of this force system remain closed, then the structure is safe against collapse along any Heymanian or non-Heymanian displacements.

Though this theorem is an interesting result from a theoretical point of view, it should be emphasized that it applies only if finding a force system which equilibrates the loads without containing any frictional components. Theorem #2, which is a refined formulation of Heyman's original Safe Theorem, has wider validity.

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